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# THE STRUCTURE OF HEREDITARY PROPERTIES AND COLOURINGS OF RANDOM GRAPHS

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We answer two fundamental questions about hereditary properties of random graphs. First, does there exist a simple and easily described property that closely approximates the hereditary property $\mathcal{P}$ in the probability space $\mathcal{G}(n,p)$? Second, does there exist a constant $c_p(\mathcal{P})$ such that the $\mathcal{P}$-chromatic number of the random graph $G_{n,p}$ is almost surely $(c_p(\mathcal{P})+o(1))n/2\log_2 n$? The second question was posed by Scheinerman (*SIAM J. Discrete Math.* **5** (1992) 74–80).

The two questions are closely related and, in the case $p=1/2$, have already been answered. Prömel and Steger (*Contemporary Mathematics* **147**, Amer. Math. Soc., Providence, 1993, pp. 167-178), Alekseev (*Discrete Math. Appl.* **3** (1993) 191-199) and the authors (*Algorithms and Combinatorics* **14** Springer-Verlag (1997) 70–78) provided an approximation which was used by the authors (*Random Structures and Algorithms* **6** (1995) 353–356) to answer the $\mathcal{P}$-chromatic question for $p=1/2$. However, the approximating properties that work well for $p=1/2$ fail completely for $p\neq 1/2$.

In this paper we describe a class of properties that do approximate $\mathcal{P}$ in $\mathcal{G}(n,p)$, in the following sense: for any desired accuracy of approximation, there is a property in our class that approximates $\mathcal{P}$ to this level of accuracy. As may be expected, our class includes the simple properties used in the case $p=1/2$.

The main difficulty in answering the second of our two questions, that concerning the $\mathcal{P}$-chromatic number of $G_{n,p}$, is that the number of small $\mathcal{P}$-graphs in $G_{n,p}$ has, in general, large variance. The variance is smaller if we replace $\mathcal{P}$ by a simple approximation, but it is still not small enough. We overcome this by considering instead a very rigid non-hereditary subproperty $\mathcal{Q}$ of the approximating property; the variance of the number of small $\mathcal{Q}$-graphs is small enough for our purpose, and the structure of $\mathcal{Q}$ is sufficiently restricted to enable us to show this by a fine analysis.

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## 1. Introduction

A *graph property* $\mathcal{P}$ (an infinite class of graphs closed under isomorphism) is said to be *hereditary* if every *induced* subgraph of every member of $\mathcal{P}$ is also in $\mathcal{P}$; a $\mathcal{P}$-*graph* is an element of $\mathcal{P}$. The property of all graphs is called the *trivial* property. For a recent survey of hereditary graph properties, see [6].

How many graphs of order $k$ have property $\mathcal{P}$? Denote by $G_{n,p}$ a random graph of order $n$, in which edges are selected independently at random with probability $p$. Let us define the constants $c_{k,p}(\mathcal{P})$, for each $k \geq 2$ and constant $0 < p < 1$, by

$$\mathbb{P}(G_{k,p} \in \mathcal{P}) = 2^{-c_{k,p}(\mathcal{P})\binom{k}{2}}.$$

Notice that, when $p=1/2$, $\mathbb{P}(G_{k,p} \in \mathcal{P})$ is exactly the number of $\mathcal{P}$-graphs of order $k$, divided by $2^{\binom{k}{2}}$. A slight extension of a result of Alekseev [1] (see also [2]) is that the limit

$$c_p(\mathcal{P}) = \lim_{k\to\infty} c_{k,p}(\mathcal{P})$$

exists if $\mathcal{P}$ is hereditary. In fact, it was further shown in [7] that for each such property $c_{k,p}(\mathcal{P})$ *increases* with $k$. In particular, $c_p(\mathcal{P}) > 0$ if $\mathcal{P}$ is non-trivial: indeed, if $c_p(\mathcal{P}) = 0$ then $c_{k,p}(\mathcal{P}) = 0$ for every $k$, so every graph is in $\mathcal{P}$.

Given integers $0 \leq s \leq r$, let $\mathcal{P}_{r,s}$ be the property comprising all graphs whose vertex sets are unions of $r$ classes, $s$ of which span complete subgraphs and the other $r-s$ spanning empty subgraphs. It is easily checked that $c_{1/2}(\mathcal{P}_{r,s}) = 1/r$. It was shown by Prömel and Steger [13] for certain properties $\mathcal{P}$, and by Alekseev [2] and by Bollobás and Thomason [8] for all properties $\mathcal{P}$, that there is some $\mathcal{P}_{r,s} \subset \mathcal{P}$ with $c_{1/2}(\mathcal{P}) = c_{1/2}(\mathcal{P}_{r,s}) = 1/r$. However, it was shown in [9] that for $p \neq 1/2$, there are properties $\mathcal{P}$ for which no $\mathcal{P}_{r,s} \subset \mathcal{P}$ can be found with $c_p(\mathcal{P}_{r,s}) = c_p(\mathcal{P})$.

The first problem that we wish to tackle in this paper is whether there exist simple properties that will approximate $\mathcal{P}$ when $p \neq 1/2$ in the same way that the properties $\mathcal{P}_{r,s}$ do when $p = 1/2$. The reason for seeking such an approximation to $\mathcal{P}$ is that it may then be possible, when working on a problem concerning $\mathcal{P}$, to replace $\mathcal{P}$ by an approximation which is much easier to handle. Our prime example of such a problem is the determination of the $\mathcal{P}$-chromatic number of random graphs, which we shall describe in detail later.

To introduce our wider class of simple properties, called *basic* properties, let us define a *type* to be a labelled graph, each of whose vertices and edges is coloured black or white. Given a type $\tau$, the basic property $\mathcal{P}(\tau)$ consists of all those graphs $G$ for which $V(G)$ has a partition $\{V_v : v \in V(\tau)\}$ *witnessing*

$\tau$; that is to say, $G[V_v]$ is complete or empty according as $v$ is black or white, and moreover, if the edge $uv$ is in $\tau$, then $G[V_u, V_v]$ is a complete or empty bipartite graph according as the edge $uv$ is black or white. Note that, if $\tau$ is an edgeless graph having $s$ black vertices and $r-s$ white vertices, then $\mathcal{P}(\tau)=\mathcal{P}_{r,s}$.

When $p=1/2$ there exists a basic property $\mathcal{P}(\tau)$ such that $\mathcal{P}(\tau)\subset\mathcal{P}$ and $c_p(\mathcal{P}(\tau))=c_p(\mathcal{P})$, namely, $\mathcal{P}(\tau)=\mathcal{P}_{r,s}$. This is no longer true when $p\neq 1/2$; we shall give a counterexample in §5.3. Nevertheless, it is true that basic properties do approximate every hereditary property in a useful way.

**Theorem 1.1.** *Let $\mathcal{P}$ be a hereditary property and let $0<\epsilon, p<1$. Then there exists a type $\tau$ such that $\mathcal{P}(\tau)\subset\mathcal{P}$ and $c_p(\mathcal{P}(\tau))\leq c_p(\mathcal{P})+\epsilon$.*

Having found highly structured properties that approximate every hereditary property, we show how they can be used to determine the $\mathcal{P}$-chromatic number of random graphs, as alluded to earlier.

A $\mathcal{P}$-*colouring* of a graph $G$ is a partition of its vertex set into some colour classes with each class inducing a $\mathcal{P}$-graph. If $\mathcal{P}$ is a hereditary property then every graph has a $\mathcal{P}$-colouring with some number of classes. The $\mathcal{P}$-*chromatic number* $\chi_{\mathcal{P}}(G)$ of $G$ is the minimal number of $\mathcal{P}$-graphs in a $\mathcal{P}$-colouring of $G$. Thus, if $\mathcal{P}$ is the property of having no edges, then $\chi_{\mathcal{P}}(G)$ is just $\chi(G)$, the usual chromatic number of $G$. Observe that $\mathcal{P}$ is trivial if and only if $\chi_{\mathcal{P}}(G)=1$ for all $G$.

In [5] it was proved that almost every random graph $G_{n,p}$ has chromatic number $(\log(1/(1-p))+o(1))n/2\log_2 n$. For every hereditary property $\mathcal{P}$, Scheinerman [14] demonstrated the existence of constants $c'_{\mathcal{P}}$ and $c''_{\mathcal{P}}$ such that $c'_{\mathcal{P}}n/\log n<\chi_{\mathcal{P}}(G_{n,p})<c''_{\mathcal{P}}n/\log n$ almost surely, and he conjectured that $\chi_{\mathcal{P}}(G_{n,p})$ is concentrated in a similar manner to $\chi(G_{n,p})$. In this paper we prove his conjecture.

**Theorem 1.2.** *Let $\mathcal{P}$ be a non-trivial hereditary graph property and let $0<p<1$. Then, almost surely,*

$$\chi_{\mathcal{P}}(G_{n,p}) = c_p\,\frac{n}{2\log_2 n}(1+o(1)),$$

*where $c_p=c_p(\mathcal{P})=\lim_{k\to\infty} c_{k,p}(\mathcal{P})$.*

As shown in [9], the case $p=1/2$ of Theorem 1.2 follows almost immediately from the result of [5] and the existence of the approximating property $\mathcal{P}_{r,s}\subset\mathcal{P}$ with $c_{1/2}(\mathcal{P}_{r,s})=c_{1/2}(\mathcal{P})$. The case $p\neq 1/2$ lies much deeper. One reason for this is the need to find good but simple approximations to $\mathcal{P}$ when $p\neq 1/2$. This need is satisfied by Theorem 1.1; indeed, given this theorem, it is clear that it is enough to establish Theorem 1.2 only for basic properties. But, even given Theorem 1.1, a substantial amount of effort and further refinement are still needed to accomplish the proof of Theorem 1.2. In particular, we shall need to approximate $\mathcal{P}(\tau)$ itself by a yet more structured property which is not even hereditary.

Before getting down to work, however, let us get out of the way the "trivial half" of Theorem 1.2, namely the lower bound for $\chi_{\mathcal{P}}(G_{n,p})$, which is an immediate consequence of the definition of $c_p(\mathcal{P})$ and its strict positivity.

**Lemma 1.1.** *Let $\mathcal{P}$ be a non-trivial hereditary graph property. Let $0<p<1$ and let $c_p=c_p(\mathcal{P})$. Then $\chi_{\mathcal{P}}(G_{n,p})\geq(1+o(1))c_p n/(2\log_2 n)$ almost surely.*

**Proof.** Let $0<\epsilon<1/8$ and let $t=\lceil(1+4\epsilon)(2/c_p)\log_2 n\rceil$. If $n$ is sufficiently large then $c_{t,p}(\mathcal{P})>(1-2\epsilon)c_p$, $t>2$ and the expected number of $\mathcal{P}$-graphs of order $t$ induced in $G_{n,p}$ is

$$\binom{n}{t}2^{-c_{t,p}(\mathcal{P})\binom{t}{2}} < \left\{n2^{-(1-2\epsilon)c_p t/2}\right\}^t < \left\{n2^{-(1+\epsilon)\log_2 n}\right\}^t = o(1).$$

Thus almost surely $G_{n,p}$ has no subgraph in $\mathcal{P}$ of order $t$, and hence $\chi_{\mathcal{P}}(G_{n,p}) \geq c_p n/(2(1+4\epsilon)\log_2 n)$. This being true for all $\epsilon>0$, the lemma follows. ∎

The remainder of the paper is structured as follows. In the next section we make some fundamental observations about all hereditary properties, and then in §3 we prove the approximation result Theorem 1.1. Following that, in §4, we turn to the $\mathcal{P}$-chromatic of random graphs and pinpoint where the difficulty lies in the proof of Theorem 1.2. We then go on in §5 to investigate basic properties in some detail, enabling us finally in §6 to complete the proof of Theorem 1.2.

It may be perceived already that, in order to define and to discuss various special properties and to describe the proofs of Theorem 1.1 and 1.2, we shall need numerous subsidiary technical definitions; rather than bury these in the text, we shall display them for the assistance of the reader.

## 2. Hereditary properties

In this section we make some basic observations about hereditary properties.

### 2.1. Limit properties.

Let $\mathcal{P}$ be a hereditary property. It may well be that there are $\mathcal{P}$-graphs which appear as subgraphs of only a finite number of other $\mathcal{P}$-graphs; we regard such graphs as inessential to $\mathcal{P}$. This prompts the following definition.

**Definition.** Let $\mathcal{P}$ be a hereditary property. The *limit* of $\mathcal{P}$, denoted $\mathcal{P}^*$, is the set of graphs in $\mathcal{P}$ with arbitrarily large extensions in $\mathcal{P}$; that is,

$$\begin{aligned}\mathcal{P}^* = \{\, H \in \mathcal{P} \,:\, &\text{for all } n \in \mathbb{N},\\ &\text{there exists } G \in \mathcal{P} \text{ with } |G| \geq n \text{ and } H \subset G \,\}.\end{aligned}$$

A property $\mathcal{P}$ is said to be a *limit* if $\mathcal{P}^* = \mathcal{P}$.

Note that $\mathcal{P}^*$ is a hereditary property. We can recover $\mathcal{P}^*$ from $\mathcal{P}$ by repeatedly removing graphs which have only finitely many extensions. A single such removal clearly will not affect the value of $c_p(\mathcal{P})$ but, on the face of it, an infinite number of removals may have a greater effect. This turns out not to be the case, as we now demonstrate.

**Theorem 2.1.** *Let $\mathcal{P}$ be a hereditary property and $\mathcal{P}^*$ its limit. Then $c_p(\mathcal{P}^*) = c_p(\mathcal{P})$.*

**Proof.** By definition, $\mathbb{P}(G_{k,p} \in \mathcal{P}) = 2^{-c_{k,p}(\mathcal{P})\binom{k}{2}}$, where $G_{k,p}$ is a random graph on vertex set $[k] = \{1, \ldots, k\}$. Recall from §1 that $c_{k,p}(\mathcal{P})$ increases to the limit $c_p(\mathcal{P})$. Now $\mathcal{P}^* \subset \mathcal{P}$ so certainly $c_p(\mathcal{P}^*) \geq c_p(\mathcal{P})$. To obtain the reverse inequality, we define the subset $\mathcal{P}|_n$ of $\mathcal{P}$ by

$$\mathcal{P}|_n = \{\, H \in \mathcal{P} \,:\, \text{ there exists } G \in \mathcal{P}, |G| \geq n \text{ and } H \subset G \,\}.$$

Then $\mathcal{P}|_n$ is a hereditary property which contains all $\mathcal{P}$-graphs of order at least $n$; in particular $c_{n,p}(\mathcal{P}|_n) = c_{n,p}(\mathcal{P})$. Moreover $\mathcal{P}^* = \bigcap_{n=1}^{\infty} \mathcal{P}|_n$.

Let $k \in \mathbb{N}$ be given. The set of graphs on vertex set $[k]$ is finite, so there exists some $n \in \mathbb{N}$ such that no $G_{k,p}$ in $\mathcal{P} - \mathcal{P}^*$ is contained in $\mathcal{P}|_n$. Therefore $c_{k,p}(\mathcal{P}|_n) \geq c_{k,p}(\mathcal{P}^*)$ (in fact, equality holds here). So, by the above remarks and monotonicity, we have

$$c_p(\mathcal{P}) \geq c_{n,p}(\mathcal{P}) = c_{n,p}(\mathcal{P}|_n) \geq c_{k,p}(\mathcal{P}|_n) \geq c_{k,p}(\mathcal{P}^*).$$

Since $c_p(\mathcal{P}^*)$ is the limit of the right hand side as $k \to \infty$, we obtain $c_p(\mathcal{P}) \geq c_p(\mathcal{P}^*)$, as claimed. ∎

It can be seen that the use of monotonicity is not essential to the above proof, which could be modified to use only the existence of the limits $c_p(\mathcal{P})$ and $c_p(\mathcal{P}^*)$. However, monotonicity makes for a much cleaner argument.

## 2.2. Irreducible properties.

Given an indexed collection of hereditary properties $\mathcal{P}_\gamma$, $\gamma \in \Gamma$, their product is defined to be

$$\prod_{\gamma\in\Gamma} \mathcal{P}_\gamma = \{ G : V(G) = \bigcup_{\gamma\in\Gamma} V_\gamma \text{ with } G[V_\gamma] \in \mathcal{P}_\gamma \}$$

where, as usual, $G[S]$ denotes the subgraph of $G$ induced by $S \subset V(G)$ (we use the notation of [3] unless otherwise stated). Thus a graph is in $\prod_{\gamma\in\Gamma}\mathcal{P}_\gamma$ if it consists of a disjoint union of some graphs from some different $\mathcal{P}_\gamma$, perhaps with extra edges added between these graphs. Observe that, since each $\mathcal{P}_\gamma$ is hereditary, so is $\prod_{\gamma\in\Gamma}\mathcal{P}_\gamma$.

**Definition.** A hereditary property is *irreducible* if it is not the product of two other hereditary properties.

According to our definition, the trivial property is not irreducible, for it equals any product one of whose factors is trivial. However, this is a nicety which does not concern us, since our interest lies in non-trivial properties.

**Lemma 2.1.** *Every non-trivial hereditary property is the product of a finite collection of irreducible hereditary properties.*

**Proof.** Let $\mathcal{P}$ be a hereditary property which cannot be expressed as a product of a finite collection of irreducible hereditary properties. Then certainly $\mathcal{P}$ is reducible, so write $\mathcal{P}$ as the product of two hereditary factors. At least one of these factors is, in turn, reducible and so can itself be written as a product, and so on. Thus, for every integer $n$, we can write $\mathcal{P} = \prod_{i=1}^n \mathcal{P}_i$ where each $\mathcal{P}_i$ is hereditary. But each $\mathcal{P}_i$ contains the graph of order 1, wherefore $\mathcal{P}$ contains every graph of order $n$. Since this is true for every $n$ it follows that $\mathcal{P}$ is trivial. ∎

We shall prove Theorem 1.2 first for irreducible properties. To extend the theorem to all properties we shall need to relate $c_p(\mathcal{P})$ to $c_p(\mathcal{P}_\gamma)$, where $\mathcal{P} = \prod_{\gamma\in\Gamma}\mathcal{P}_\gamma$.

**Theorem 2.2.** *Let $\mathcal{P}_1, \ldots, \mathcal{P}_k$ be non-trivial hereditary properties and let $\mathcal{P} = \mathcal{P}_1 \times \ldots \times \mathcal{P}_k$. Then*

$$\frac{1}{c_p(\mathcal{P})} = \sum_{i=1}^{k} \frac{1}{c_p(\mathcal{P}_i)}.$$

**Proof.** Let $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$. Let $n \in \mathbb{N}$ be fixed and let $\beta = (\beta_1, \ldots, \beta_k) \in (\mathbb{N}_0/n)^k$ be a vector for which the expression $\prod_{i=1}^k 2^{-c_{n,p}(\mathcal{P}_i)\binom{\beta_i n}{2}}$ attains its maximum,

$M_n$, subject to $\sum \beta_i = 1$. By considering a fixed partition of $[n]$ into $k$ parts with $\beta_1 n, \ldots, \beta_k n$ vertices, we see that $\mathbb{P}(G_{n,p} \in \mathcal{P}) \geq M_n$. On the other hand, there are at most $k^n$ partitions of $[n]$ into $k$ parts, so $\mathbb{P}(G_{n,p} \in \mathcal{P}) \leq k^n M_n$. Therefore

$$-n \log_2 k + \sum_{i=1}^{k} \binom{\beta_i n}{2} c_{n,p}(\mathcal{P}_i) \leq c_{n,p}(\mathcal{P}) \binom{n}{2} \leq \sum_{i=1}^{k} \binom{\beta_i n}{2} c_{n,p}(\mathcal{P}_i).$$

Now let $M$ be the maximum of the expression $\prod_{i=1}^{k} 2^{-c_p(\mathcal{P}_i)\beta_i^2}$, taken over all $\beta = (\beta_1, \ldots, \beta_k) \in [0,1]^k$ such that $\sum \beta_i = 1$. It is easily seen that $M_n^{-\binom{n}{2}} \to M$ as $n \to \infty$. Therefore it follows from the inequalities above that

$$c_p(\mathcal{P}) = \sum_{i=1}^{k} \beta_i^2 c_p(\mathcal{P}_i),$$

where $\beta$ is chosen to minimize the right hand side subject to the constraint $\sum \beta_i = 1$. Now each $c_p(\mathcal{P}_i)$ is non-zero because $\mathcal{P}_i$ is non-trivial. So at this minimum value $\beta_i c_p(\mathcal{P}_i)$ is independent of $i$. Since $\sum \beta_i = 1$ this means that $\beta_i = 1/s c_p(\mathcal{P}_i)$, where $s = \sum_{i=1}^{k} 1/c_p(\mathcal{P}_i)$. Hence $c_p(\mathcal{P}) = \sum_{i=1}^{k} \beta_i(\beta_i c_p(\mathcal{P}_i)) = \sum \beta_i / s = 1/s$, as claimed. ∎

We are now able to show that we need consider only irreducible properties from this point on.

**Corollary 2.1.** *Let $0 < p < 1$ be fixed and suppose that Theorem 1.2 holds for each of the hereditary properties $\mathcal{P}_1, \ldots, \mathcal{P}_k$. Then the theorem holds for the property $\mathcal{P} = \prod_{i=1}^{k} \mathcal{P}_i$.*

**Proof.** Let $s = \sum_{i=1}^{k} 1/c_i$, where $c_i = c_p(\mathcal{P}_i)$. Partition $V(G_{n,p})$ into $k$ sets $V_1, \ldots, V_k$ with $|V_i| = n_i \geq \lfloor n/(s c_i) \rfloor$, $1 \leq i \leq k$. By applying Theorem 1.2 to the random graph $G_{n,p}[V_i]$ of order $n_i$ we see that $V_i$ can, almost surely, be partitioned into a collection of $(1+o(1)) c_i n_i / (2 \log_2 n_i) = (1+o(1)) n / (2s \log_2 n)$ sets each inducing a $\mathcal{P}_i$-graph. By adding empty sets to the partition of $V_i$, if necessary, we may assume that each $V_i$ is partitioned into the same number of sets. By forming groups of $k$ sets, one from each partition, we see that, almost surely, $V(G_{n,p})$ can be partitioned into $(1+o(1)) n / (2s \log_2 n)$ sets each inducing a $\mathcal{P}$-graph. By Lemma 1.1 and Theorem 2.2 this means that Theorem 1.2 holds for $\mathcal{P}$. ∎

## 3. Approximation by basic properties

Let us remind the reader of the definition of a basic property.

**Definition.** A *type* is a labelled graph, each of whose vertices and edges is coloured black or white. The set of black vertices of a type $\tau$ is denoted by $B(\tau)$, the white vertices by $W(\tau)$, the set of black edges by $EB(\tau)$ and the set of white edges by $EW(\tau)$.

The *basic property* $\mathcal{P}(\tau)$ consists of all those graphs $G$ for which $V(G)$ has a partition $\{V_v : v \in V(\tau)\}$ *witnessing* $\tau$; that is to say, $G[V_v]$ is complete or empty according as $v \in B(\tau)$ or $v \in W(\tau)$, and if the edge $uv$ is in $\tau$, then $G[V_u, V_v]$ is a complete or empty bipartite graph according as $uv \in EB(\tau)$ or $uv \in EW(\tau)$.

As discussed in the introduction, we know that when $p = 1/2$, for every hereditary property $\mathcal{P}$ there is a basic property $\tau = \tau(p)$ such that $\mathcal{P}(\tau) \subset \mathcal{P}$ and $c_p(\mathcal{P}(\tau)) = c_p(\mathcal{P})$, namely some property $\mathcal{P}_{r,s}$. This fails to be true for $p \neq 1/2$, though we delay the discussion of a counterexample to §5.3 when we shall be better equipped to analyze it.

Nevertheless, our purpose in this section is to demonstrate that, although there need not exist a $\tau$ with $\mathcal{P}(\tau) \subset \mathcal{P}$ and $c_p(\mathcal{P}(\tau)) = c_p(\mathcal{P})$, all is not lost. We shall prove Theorem 1.1, showing that it is always possible for us to find a sequence $(\tau_n)_{n=1}^{\infty}$ such that $\mathcal{P}(\tau_n) \subset \mathcal{P}$ and $c_p(\mathcal{P}(\tau_n)) \to c_p(\mathcal{P})$ as $n \to \infty$. For the purpose of proving Theorem 1.2, this degree of approximation will be sufficient.

We begin with three important definitions.

**Definition.** A *prototype* is a labelled graph, some of whose edges are coloured black or white, the remaining edges being uncoloured. The uncoloured edges of $\pi$ are denoted by $EU(\pi)$. A type $\tau$ is said to *extend* a prototype $\pi$ if $V(\pi) = V(\tau)$, $E(\pi) = E(\tau)$, $EB(\pi) \subset EB(\tau)$ and $EW(\pi) \subset EW(\tau)$. In other words, $\tau$ is formed from $\pi$ by colouring its vertices and the edges of $EU(\pi)$.

**Definition.** Let $H$ be a graph with a specified vertex partition $V(H) = U_1 \cup \ldots \cup U_k$. For $1 \le i < j \le k$ let $E_{ij}(H) = \{uv : u \in U_i \text{ and } v \in U_j\}$. Let $\pi$ be a prototype with vertex set $[k]$. Then $\pi \circ H$ is the set of graphs $F$ with $V(F) = V(H)$, such that

$$E_{ij}(F) = \begin{cases} E_{ij}(H) & \text{if } ij \notin E(\pi), \\ U_i \times U_j & \text{if } ij \in EB(\pi), \\ \emptyset & \text{if } ij \in EW(\pi). \end{cases}$$

Note that in the unlisted fourth case, namely when $ij \in EU(\pi)$, the sets $E_{ij}(F)$ are not constrained at all. In other words, the graphs in $\pi \circ H$ are those that can be obtained from $H$ by making the bipartite subgraph induced by $U_i$ and $U_j$ complete if $ij$ is black, empty if $ij$ is white, anything at all if $ij$ is an uncoloured edge, and by placing any subgraphs whatsoever within the classes $U_i$. So for example, setting $M=\sum_{i=1}^{|\pi|}\binom{|U_i|}{2}+\sum_{ij\in EU(\pi)}|U_i||U_j|$, we have $|\pi\circ H|=2^M$.

**Definition.** Let $k,\ell\in\mathbb{N}$. A $(k,\ell)$-*universal* graph is a graph $H$, equipped with a vertex partition $U_1\cup\cdots\cup U_k$, where $|U_1|=\ldots=|U_k|$, having the following property. Let $\pi$ be any prototype with vertex set $[k]$ and let $F\in\pi\circ H$. Then there is a type $\tau$ extending $\pi$, such that every graph of order $\ell$ representing $\tau$ is an induced subgraph of $F$.

The universal graphs are the vital tools that will enable us to reduce colouring by arbitrary hereditary properties to colouring by basic properties.

**Lemma 3.1.** *Let $k,\ell\in\mathbb{N}$. Then there exists a $(k,\ell)$-universal graph.*

**Proof.** The proof is a relatively standard exercise in induced Ramsey theory, and there are several ways that a construction can be exhibited. The following is based on the amalgamation method of Nešetřil and Rödl.

Consider all $k$-partite graphs with vertex partition $V_1,\ldots,V_k$ where $|V_i|=\ell$, $1\le i\le\ell$. There are $2^{\binom{k}{2}\ell^2}$ such graphs. Let $J_0$ be the vertex disjoint union of all these graphs; $J_0$ has a natural partition into $k$-parts, its $i^{\text{th}}$ part being the union of the $i^{\text{th}}$ parts of the $2^{\binom{k}{2}\ell^2}$ constituent graphs.

Now construct, successively, $k$-partite graphs $J_1,\ldots,J_k$ as follows. To form $J_i$, let $r$ be the number of vertices in the $i^{\text{th}}$ part of $J_{i-1}$ and let $X$ be a set of $R=R(r)$ vertices, where $R(r)$ is the Ramsey number of $r$ (to be exact, the smallest number such that every graph of order $|X|$ contains either a complete subgraph of order $r$ or an independent set of order $r$). Now take $\binom{R}{r}$ vertex disjoint copies of $J_{i-1}$, one for each $r$-subset $Y\in X^{(r)}$; label them $J_{i-1}^Y$, $Y\in X^{(r)}$. Then identify the $i^{\text{th}}$ part of $J_{i-1}^Y$ with $Y$. The resulting graph is $J_i$; it is naturally $k$-partite, its $i^{\text{th}}$ part being $X$ and its $j^{\text{th}}$ part being the disjoint union of the $j^{\text{th}}$ parts of the $J_{i-1}^Y$ for $j\ne i$.

We now form, in a similar manner, a sequence of graphs $H_0,\ldots,H_{\binom{k}{2}}$. To begin with, let $H_0=J_k$. Let $K_k$ be the complete graph on vertex set $[k]$ and order the edges of $K_k$ in some way. To form $H_m$, let $ij$ be the $m^{\text{th}}$ edge of $K_k$ in the chosen ordering. Let $s$ and $t$ be the cardinalities of the $i^{\text{th}}$ and $j^{\text{th}}$ parts of $H_{m-1}$. Let $X$ and $Z$ be two disjoint sets of $R=R(s,t)$ vertices, where $R(s,t)$ is the bipartite Ramsey number for the pair $(s,t)$. (That is to say, every bipartite graph with vertex classes $X$ and $Z$ will contain either a complete bipartite subgraph $K_{s,t}$ or an empty bipartite subgraph with $s$ vertices in $X$ and $t$ in $Z$; for information on the numbers $R(s,t)$ see [18].)

Now take $\binom{R}{s}\binom{R}{t}$ vertex disjoint copies of $H_{m-1}$, one for each pair $Y \in X^{(s)}$ and $W \in Z^{(t)}$; label them $H_{m-1}^{YW}$, $Y \in X^{(s)}$, $W \in Z^{(t)}$. Then identify the $i^{\text{th}}$ part of $M_{m-1}^{YW}$ with $Y$ and the $j^{\text{th}}$ part with $W$. The resulting graph is $H_m$, and it is naturally $k$-partite, its $i^{\text{th}}$ and $j^{\text{th}}$ parts being $X$ and $Z$ and its other parts being the disjoint union of the corresponding parts of the graphs $H_{m-1}^{YW}$.

The final graph formed in this way is $H_{\binom{k}{2}}$. Let $H$ be obtained from this graph by adding isolated vertices as necessary so that the parts of $H$ are of equal size. It is not hard to verify that $H$ is $(k,\ell)$-universal. For let $\pi$ be some prototype on vertex set $[k]$ and let $F \in \pi \circ H$. We colour $\pi$ to form a type $\tau$ as follows. Let the $m^{\text{th}}$ edge of $K_k$ be the last one appearing in $EU(\pi)$. By the construction of $H_m$, $F$ contains a copy of $H_{m-1}$ in which the bipartite subgraph spanned by the classes corresponding to the $m^{\text{th}}$ edge is made either complete or empty. Colour this edge of $EU(\pi)$ black or white accordingly and call the resulting prototype $\pi_m$; then $F$ contains an induced subgraph in the class $\pi_m \circ H_{m-1}$. Now let the penultimate edge of $K_k$ appearing in $EU(\pi)$ have label $m'$. Then, by the construction of $H_{m'}$, $F$ contains a copy of $H_{m'-1}$ in which the bipartite subgraph spanned by the classes corresponding to edge $m'$ is made either complete or empty. Colouring this edge of $EU(\pi_m)$ appropriately, to form the prototype $\pi_{m'}$, we see that $F$ contains an induced subgraph in the class $\pi_{m'} \circ H_{m'-1}$. Proceeding in this manner we colour all of $EU(\pi)$. If $\pi_0$ is the prototype produced, it follows that $F$ contains an induced subgraph $F_0 \in \pi_0 \circ H_0$.

Now $J_k = H_0$ and, by the definition of $J_k$, $F_0$ contains a copy of $J_{k-1}$ whose $k^{\text{th}}$ part spans a complete or empty subgraph; colour vertex $k$ of $\pi_0$ black or white as the case may be. Carrying on, we may colour all the vertices of $\pi_0$ and this gives us a type $\tau$. This type has the property that $F$ contains an induced subgraph $F_1 \in \tau \circ J_0$ such that a class of $F_1$ is complete or empty according as the corresponding vertex of $\tau$ is black or white. But then, by the definition of $J_0$, $F_1$ clearly contains as an induced subgraph every graph representing $\tau$ with up to $\ell$ vertices in each class and, in particular, every such graph of order $\ell$. ∎

We shall need to make use of the well-known lemma of Szemerédi [17]. A pair of subsets $U$ and $W$ of the vertex set of a graph $G$ is said to be $\eta$-*regular* if $|d(U,W) - d(U',W')| < \eta$ whenever $U' \subseteq U$, $|U'| > \eta|U|$ and $W' \subseteq W$, $|W'| > \eta|W|$, where $d(U,W) = e(U,W)/|U||W|$. Let $\mathcal{V} = \{V_1, \dots, V_k\}$ be a partition of the vertex set of $G$ into $k$ non-empty parts $V_1, \dots, V_k$. The *order* of the partition is $|\mathcal{V}| = k$, and if $\lfloor |G|/|\mathcal{V}| \rfloor \le |V_i| \le \lceil |G|/|\mathcal{V}| \rceil$ for each $V_i \in \mathcal{V}$ then we call the partition *equitable*. For a non-empty closed interval of real numbers $[a,b]\subset\mathbb{R}$, we define

$$\mathcal{V}_G[a,b]=\left\{ij\in[|\mathcal{V}|]^{(2)} : (V_i,V_j) \text{ is } \eta\text{-regular and } d(V_i,V_j)\in[a,b]\right\}.$$

The notation extends to half-open intervals, so that, for example, $\mathcal{V}_G(a,b]$ has the obvious meaning. We also define

$$\mathcal{V}_G\emptyset=\left\{ij\in[|\mathcal{V}|]^{(2)} : (V_i,V_j) \text{ is not } \eta\text{-regular}\right\}.$$

Given this terminology, Szemerédi's Regularity Lemma is (equivalent to) the statement that, given $\eta>0$ and an integer $k$, there is an integer $L=L(k,\eta)$ such that every graph $G$ of order at least $k$ has an equitable partition $\mathcal{V}$ satisfying both $k\le|\mathcal{V}|\le L$ and $|\mathcal{V}_G\emptyset|\le\eta\binom{|\mathcal{V}|}{2}$. We call such a partition a *Szemerédi partition* of the graph.

Prototypes afford a convenient way to describe a partition of a graph.

**Definition.** Let $0<\lambda<1$ and let $\pi$ be a prototype. We say that the graph $G$ *conforms to* $\pi$ with respect to $\lambda$ and $\eta$ if $G$ has an equitable partition $\mathcal{V}$ into $|\pi|$ parts, such that

$$EB(\pi)=\mathcal{V}_G(1-\lambda,1],\qquad EW(\pi)=\mathcal{V}_G[0,\lambda)\qquad\text{and}\qquad EU(\pi)=\mathcal{V}_G\emptyset.$$

A standard application of regularity is the next lemma, found in [8, Lemma 3]; similar results, along with many extensions, can be found in the survey of Komlós and Simonovits [11].

**Lemma 3.2.** *Let $\pi$ be a prototype with vertex set $[k]$. Let $0<\lambda,\eta<1$ satisfy $k\eta\le\lambda^{k-1}$. Suppose that $G$ is a graph with a partition $\mathcal{V}=\{V_1,\dots,V_k\}$ of order $k$, satisfying $EB(\pi)\subset\mathcal{V}_G[\lambda,1]$ and $EW(\pi)\subset\mathcal{V}_G[0,1-\lambda]$. Then there exist vertices $v_i\in V_i$, $1\le i\le k$, such that $v_iv_j\in E(G)$ if $ij\in EB(\tau)$ and $v_iv_j\notin E(G)$ if $ij\in EW(\tau)$.* ∎

The following lemma is crucial to our proof of Theorem 1.1.

**Lemma 3.3.** *Let $\ell,k\in\mathbb{N}$ and let $0<\lambda<1$. Then there exists $\eta_0=\eta_0(\ell,k,\lambda)>0$ and $n_0=n_0(\ell,k)\in\mathbb{N}$ such that the following holds. Let $\pi$ be a prototype with $|\pi|=k$, let $0<\eta\le\eta_0$ and let $G$ be a graph of order $|G|\ge n_0$ conforming to $\pi$ with respect to $\lambda$ and $\eta$. Then there is a type $\tau$ extending $\pi$, such that $G$ contains as an induced subgraph every graph of order $\ell$ representing $\tau$.*

**Proof.** Let $H$ be a $(k,\ell)$-universal graph; Lemma 3.1 tells us that such an $H$ exists. By the definition of a universal graph, it is enough to find an induced subgraph $F$ of $G$ such that $F\in\pi\circ H$.

Let the partition of $V(H)$ be $\{U_1,\dots,U_k\}$ where $|U_i|=u$ for $1\le i\le k$. Let $U_i=\{u_i^a : 1\le a\le u\}$. Define a prototype $\rho$ with vertex set $V(H)$ as follows. If $ij\in EB(\pi)$, then $u_i^a u_j^b\in EB(\rho)$ for $1\le a,b\le u$. If $ij\in EW(\pi)$, then $u_i^a u_j^b\in EW(\rho)$ for $1\le a,b\le u$. If $ij\notin E(\pi)$, then $u_i^a u_j^b\in EB(\rho)$ if $u_i^a u_j^b\in E(H)$ and $u_i^a u_j^b\in EW(\rho)$ if $u_i^a u_j^b\notin E(H)$. There are no other edges in $\rho$. Then $G$ contains an induced subgraph in $\pi\circ H$ if and only if there exist vertices $v_i^a\in G$, $1\le a\le u$, $1\le i\le k$, such that $v_i^a v_j^b\in E(G)$ whenever $u_i^a u_j^b\in EB(\rho)$ and $v_i^a v_j^b\notin E(G)$ whenever $u_i^a u_j^b\in EW(\rho)$.

We define $n_0=2ku$, $\mu=\lambda/2$, $\nu=\mu^{ku-1}/ku$ and $\eta_0=\nu/2u$. Let $V_1,\dots,V_k$ be the Szemerédi partition of $G$ showing that $G$ conforms to $\pi$ with respect to $\lambda$ and some $\eta\le\eta_0$. Partition $V_i$ equitably into $u$ disjoint sets $V_i^a$, $1\le a\le u$; then the sets $V_i^a$ are non-empty provided $|G|\ge n_0$. Since $|V_i^a|\ge\lfloor|V_i|/u\rfloor>\eta|V_i|$ we have $|d(V_i^a,V_j^b)-d(V_i,V_j)|<\eta$. Moreover if $U\subset V_i^a$ with $|U|>\nu|V_i^a|$ then $|U|>\nu|V_i|/2u\ge\eta|V_i|$. So, if also $W\subset V_j^b$ with $|W|>\nu|V_j^b|$, then $|d(U,W)-d(V_i,V_j)|<\eta$, and so $|d(U,W)-d(V_i^a,V_j^b)|<2\eta<\nu$. In particular, the pair $(V_i^a,V_j^b)$ is $\nu$-regular whenever the pair $(V_i,V_j)$ is $\eta$-regular.

Apply Lemma 3.2 to $G$ with the partition $\mathcal{V}=\{V_i^a : 1\le a\le u, 1\le i\le k\}$, the prototype $\rho$ instead of $\pi$, and with $\mu$ and $\nu$ in place of $\lambda$ and $\eta$. Observe that, if $u_i^a u_j^b\in EB(\rho)$, then either $ij\notin E(\pi)$ or $ij\in EB(\pi)$, so $d(V_i^a,V_j^b)>d(V_i,V_j)-\eta\ge\lambda-\eta>\mu$ since $\eta<\nu<\mu=\lambda/2$. Hence $(V_i^a,V_j^b)\in\mathcal{V}_G[\mu,1]$. Likewise, if $u_i^a u_j^b\in EW(\rho)$ then $(V_i^a,V_j^b)\in\mathcal{V}_G[0,1-\mu]$. Therefore the conditions of Lemma 3.2 are satisfied. Hence there exist vertices $v_i^a\in G$, $1\le a\le u$, $1\le i\le k$, such that $v_i^a v_j^b\in E(G)$ whenever $u_i^a u_j^b\in EB(\rho)$ and $v_i^a v_j^b\notin E(G)$ whenever $u_i^a u_j^b\in EW(\rho)$. As pointed out above, this is enough to complete the proof. ∎

We are now ready to prove our first main result, namely, Theorem 1.1.

**Proof of Theorem 1.1.** As usual, let $q=1-p$. Let $R=\max\{1/p,1/q\}$ and let $k=1+\lceil(5/\epsilon)\log_2 R\rceil$. We shall show that, for each $t\in\mathbb{N}$, there exists a type $\tau=\tau(t)$ such that $|\tau|=k$, every graph in $\mathcal{P}(\tau)$ of order $\ell=tk$ is also in $\mathcal{P}$, and $c_{\ell,p}(\mathcal{P}(\tau))\le c_p(\mathcal{P})+\epsilon$. This is enough to prove the theorem, because there are only finitely many $\tau$ with $|\tau|=k$, so there is some $\tau$ which arises as $\tau(t)$ for infinitely many $t$, implying both $\mathcal{P}(\tau)\subset\mathcal{P}$ and $c_p(\mathcal{P}(\tau))\le c_p(\mathcal{P})+\epsilon$.

We shall need a handful of constants that are “sufficiently large” or “sufficiently small”. It is quite easy to produce a very plausible but incorrect proof of the theorem in which these constants depend on each other in a circular way. For this reason the following list of definitions is given, even though it may appear pedestrian.

Let $k$ be as defined above, let $t \in \mathbb{N}$ be given and let $\ell = tk$. Let $\lambda \leq 1/2R$ be chosen so that $R^{2\lambda}(e/\lambda)^\lambda < 2^{-\epsilon/6}$. Let $\eta = \min\{\eta_0(\ell,k,\lambda), \epsilon/(5\log_2 R)\}$, where $\eta_0$ is given by Lemma 3.3, and let $L = L(k,\eta)$ be the constant given by Szemerédi's lemma. Choose $n_1$ larger than the constant $n_0(\ell,k)$ given by Lemma 3.3 and large enough so that $c_{n,p}(\mathcal{P}) < c_p(\mathcal{P}) + \epsilon/5$ for all $n \geq n_1$. The only variable we do not specify explicitly is $n$; at various points throughout the proof we shall use the phrase "if $n$ is large enough" to mean that $n$ is larger than some suitable function of $\epsilon$, $R$, $k$, $\ell$, $\lambda$, $\eta$ and $L$. It is always assumed that $n \geq n_1$.

By Szemerédi's lemma, each graph of order $n$ in $\mathcal{P}$ conforms, with respect to $\lambda$ and $\eta$, to some prototype $\pi$ with $k \leq |\pi| < L$ and $|EU(\pi)| \leq \eta\binom{|\pi|}{2}$. There being a bounded number of such prototypes, it follows that there is some prototype $\pi$ for which

$$\mathbb{P}\{G_{n,p} \in \mathcal{P} \text{ and } G_{n,p} \text{ conforms to } \pi\ \} \geq 2^{-(c_{n,p}(\mathcal{P})+\epsilon/5)\binom{n}{2}} \geq 2^{-(c_p+2\epsilon/5)\binom{n}{2}}$$

provided $n$ is sufficiently large where, as usual, we are writing $c_p$ for $c_p(\mathcal{P})$. Note that $\pi$ may depend on $n$. We shall show that $c(\pi) \leq c_p + 4\epsilon/5$ where, for any prototype $\rho$, the quantity $c(\rho)$ is defined by

$$c(\rho) = \binom{|\rho|}{2}^{-1}[\,|EB(\rho)|\log_2(1/p) + |EW(\rho)|\log_2(1/q) + |EU(\rho)|\log_2 R\,].$$

Recall that for the prototype $\pi$ we have $|EU(\pi)| \leq \eta\binom{|\pi|}{2}$, and so

$$\begin{aligned} |EB(\pi)|\log_2(1/p) + |EW(\pi)|\log_2(1/q) &\geq (c(\pi) - \eta\log_2 R)\binom{|\pi|}{2} \\ &\geq (c(\pi) - \epsilon/5)\binom{|\pi|}{2}. \end{aligned}$$

A random graph $G_{n,p}$ on vertex set $[n]$ has at most $|\pi|^n$ partitions $V_1 \cup \ldots \cup V_{|\pi|}$ with $N-1 \leq |V_i| \leq N$, where $N = \lceil n/|\pi| \rceil$. If $G_{n,p}$ conforms to $\pi$ then one of these partitions satisfies $d(V_i,V_j) \geq 1-\lambda$ for every $ij \in EB(\pi)$ and $d(V_i,V_j) \leq \lambda$ for every $ij \in EW(\pi)$. Therefore the probability that $G_{n,p}$ conforms to $\pi$ is at most

$$L^n \times \left(\sum_{i=0}^{\lambda N^2}\binom{N^2}{i}q^i p^{(N-1)^2-i}\right)^{|EB(\pi)|} \times \left(\sum_{i=0}^{\lambda N^2}\binom{N^2}{i}p^i q^{(N-1)^2-i}\right)^{|EW(\pi)|}.$$

It is easily checked that, since $\lambda \le 1/2R$, each term in these sums is at least double the preceding term. Thus the probability in question is bounded by

$$L^n \left[ 2\binom{N^2}{\lambda N^2} \left(\frac{q}{p}\right)^{\lambda N^2} p^{(N-1)^2} \right]^{|EB(\pi)|} \left[ 2\binom{N^2}{\lambda N^2} \left(\frac{p}{q}\right)^{\lambda N^2} q^{(N-1)^2} \right]^{|EW(\pi)|}$$

$$\le L^n \left[ 2R^{2N+\lambda N^2} \binom{N^2}{\lambda N^2} \right]^{\binom{|\pi|}{2}} p^{|EB(\pi)|N^2} q^{|EW(\pi)|N^2}$$

$$\le \left[ R^{2\lambda} \left(\frac{e}{\lambda}\right)^{\lambda} \right]^{\binom{|\pi|}{2} N^2} 2^{-(|EB(\pi)| \log_2(1/p) + |EW(\pi)| \log_2(1/q))N^2}$$

$$\le 2^{(\epsilon/5)\binom{n}{2}} 2^{-(c(\pi)-\epsilon/5)\binom{n}{2}}$$

provided $n$ is large enough. We now see that

$$2^{-(c_p+2\epsilon/5)\binom{n}{2}} \le \mathbb{P}\{G_{n,p} \text{ conforms to } \pi\} \le 2^{-(c(\pi)-2\epsilon/5)\binom{n}{2}},$$

which yields the promised inequality $c(\pi) \le c_p + 4\epsilon/5$.

Consider now all the $\binom{|\pi|}{k}$ prototypes induced within $\pi$ by vertex subsets of order $k$. Since the equation $c(\pi) = \binom{|\pi|}{k}^{-1} \sum_{\rho \subset \pi} c(\rho)$ holds, there must be some prototype induced in $\pi$, call it $\sigma$, such that $|\sigma| = k$ and $c(\sigma) \le c(\pi)$. Take a graph $G_{n,p} \in \mathcal{P}$ which conforms to $\pi$ and let $G_\sigma$ be the subgraph of $G_{n,p}$ induced by the vertices in $\bigcup_{i\in\sigma} V_i$. By Lemma 3.3 applied to the graph $G_\sigma$ and the prototype $\sigma$, there is a type $\tau$ extending $\sigma$ such that every graph of order $\ell$ representing $\tau$ is an induced subgraph of $G_\sigma$, and hence is in $\mathcal{P}$.

All that remains, then, is to verify that $c_{\ell,p}(\mathcal{P}(\tau)) \le c_p(\mathcal{P}) + \epsilon$. But this is straightforward. For consider a fixed partition of $[\ell]$ into $k$ parts each of size $t = \ell/k$. The probability that a random graph on vertex set $[\ell]$ witnesses $\tau$ via this partition is

$$p^{|B(\tau)|\binom{t}{2}} q^{|W(\tau)|\binom{t}{2}} p^{|EB(\tau)|t^2} q^{|EW(\tau)|t^2}$$
$$\ge 2^{-((k/2)\log_2 R + |EB(\tau)| \log_2(1/p) + |EW(\tau)| \log_2(1/q))t^2}.$$

Now $\tau$ is a colouring of the vertices and the uncoloured edges of $\sigma$. So

$$(k/2)\log_2 R + |EB(\tau)| \log_2(1/p) + |EW(\tau)| \log_2(1/q)$$
$$\le (k/2)\log_2 R + c(\sigma)\binom{k}{2},$$

and by the definition of $k$ we have $\log_2 R \le (\epsilon/5)(k-1)$. Therefore

$$2^{-(c(\sigma)+\epsilon/5)\binom{k}{2}t^2} \le \mathbb{P}\{G_{\ell,p} \in \mathcal{P}(\tau)\} = 2^{-c_{\ell,p}(\mathcal{P}(\tau))\binom{\ell}{2}}.$$

Hence $c_{\ell,p}(\mathcal{P}(\tau)) \le c(\sigma) + \epsilon/5 \le c(\pi) + \epsilon/5 \le c_p(\mathcal{P}) + \epsilon$. ∎

## 4. Colouring random graphs

We now begin the task of proving Theorem 1.2. Following [5], our aim is to obtain a very sharp estimate of the probability that a random graph contains a large $\mathcal{P}$-subgraph. Given such an estimate, Theorem 1.2 will follow because of the next lemma.

**Lemma 4.1.** *Let $\mathcal{P}$ be a hereditary graph property and let $0 < p < 1$. Suppose that, for every $0 < \epsilon < 1/3$, there exists $\delta > 0$ such that, if $n$ is sufficiently large, then with probability greater than $1 - 2^{-n^{1+\delta}}$ a random graph $G_{n,p}$ contains an induced $\mathcal{P}$-graph of order $\lfloor (2/c_p(\mathcal{P}) - \epsilon) \log_2 n \rfloor$. Then Theorem 1.2 holds for this $\mathcal{P}$ and this $p$.*

**Proof.** Let $\epsilon$ and $\delta$ be as given by the lemma. Let $t = \lceil (2/c_p(\mathcal{P}) - 2\epsilon) \log_2 n \rceil$ and let $m = \lfloor \epsilon n/t \rfloor$. If $n$ is sufficiently large then $t < \lfloor (2/c_p(\mathcal{P}) - \epsilon) \log_2 m \rfloor$. Hence, if $n$ is large, the probability that $G_{n,p}$ contains a set of $m$ vertices amongst which there is no induced $\mathcal{P}$-graph of order $t$ is at most $\binom{n}{m} 2^{-m^{1+\delta}} \leq 2^n 2^{-m^{1+\delta}} = o(1)$.

So, almost surely, there exists in $G_{n,p}$ a collection of disjoint $t$-sets whose union has order at least $n - m$ such that each $t$-set induces a $\mathcal{P}$-graph, implying that $\chi_{\mathcal{P}}(G_{n,p}) \leq (n-m)/t + m < (1 + 3\epsilon) c_p n/(2 \log_2 n)$. This being the case for all $0 < \epsilon < 1/3$, it follows that $\chi_{\mathcal{P}}(G_{n,p}) \leq (1 + o(1)) c_p n/(2 \log_2 n)$ almost surely. Combined with Lemma 1.1, this inequality completes the proof. ∎

Let us denote by $Y = Y(\mathcal{P}, n, p, t)$ the random variable counting the number of $t$-subsets of $G_{n,p}$ that induce a $\mathcal{P}$-graph. If we are to make use of Lemma 4.1, we must show that $\mathbb{P}\{Y = 0\}$ is extremely small when $t$ is only slightly smaller than $(2/c_p) \log_2 n$. In [5], where Theorem 1.2 is proved for the property of having no edges, an estimate like that of Lemma 4.1 was obtained by defining a suitable martingale and applying the Azuma-Hoeffding inequality (for a survey, see McDiarmid [12]). Other options for this case are to use Janson's inequality [10] or an inequality of Talagrand (see Steele [15] for an account).

It seems to us that, in order to apply any of these powerful inequalities to a proof of Lemma 4.1, it is necessary to estimate the second moment of $Y$. A proof of the lemma based on the martingale approach of [5] requires an estimate of the mean, not just of $Y$, but of the variable $X$ which is the maximum number of edge-disjoint $t$-subsets inducing $\mathcal{P}$-graphs. The mean $\mathbb{E}(X)$ can be related to $\mathbb{E}(Y)$ by estimating the second moment of $Y$; if it is small then $\mathbb{E}(X)$ and $\mathbb{E}(Y)$ are roughly equal. Similarly, to apply Janson's inequality it is necessary to know the value of (an expression closely akin to) the second moment of $Y$. In fact, Janson's inequality itself works in the present context only for *monotone* properties $\mathcal{P}$ (those closed under the removal of edges as well as vertices). The cognate inequality of Suen [16] will work for a general hereditary property, but the same remarks about the second moment still apply. Talagrand's inequality applies very generally but shows that $X$ is concentrated near its median, rather than its mean. Its use would require us to show that the median of $X$ lies substantially above zero. Since the maximum possible value of $X$ is large it is conceivable that the median could be quite small; showing that this is not the case more or less requires us once again to estimate the second moment of $Y$.

Hence we find ourselves driven to make some kind of estimate of the second moment of the number of subgraphs in $G_{n,p}$ having $\mathcal{P}$. However, it turns out that this leads to great difficulties that are not present in the case of the ordinary chromatic number. The computation of the variance of $Y$ involves finding the probabilities of pairs of $\mathcal{P}$-graphs overlapping in some given way, and summing over all possible ways of overlapping. Now if $\mathcal{P}$ is the property of being edgeless, as in [5], this calculation is straightforward. But in general the calculation is more or less hopeless. The probability that two $\mathcal{P}$-subgraphs overlap by a certain number of vertices depends to an enormous and uncontrollable extent on which particular $\mathcal{P}$-graphs are involved and on the way they intersect.

It is for this reason that we restrict our attention to very highly structured properties, namely basic properties. In fact, even the structure of basic properties is not quite rigid enough, and we shall need to restrict slightly further (to irreducible $p$-core constrained properties) before we are able to obtain estimates on the variance that are good enough. The details will be given in the next section.

## 5. Basic and core properties

We now begin a more detailed examination of basic properties.

### 5.1. Basic properties.

It is convenient to make the following definition.

**Definition.** Let $\tau$ be a type. Let $\mathcal{V} = \{V_v : v \in V(\tau)\}$ be a collection of pairwise disjoint finite sets indexed by $V(\tau)$; thus $\mathcal{V}$ is a partition of $V = \bigcup_{v\in V(\tau)} V_v$. Let $\mathcal{P}(\tau, \mathcal{V})$ be the set of all graphs $G$ with vertex set $V$

such that $G[V_v]$ is complete or empty according as $v$ is black or white, and moreover, if the edge $uv$ is in $\tau$, then $G[V_u, V_v]$ is a complete or empty bipartite graph according as the edge $uv$ is black or white.

Thus the basic property $\mathcal{P}(\tau)$ is $\bigcup_{\mathcal{V}} \mathcal{P}(\tau, \mathcal{V})$, and if $G \in \mathcal{P}(\tau, \mathcal{V})$ we say that $\mathcal{V}$ is a *witness* to $G \in \mathcal{P}(\tau)$.

Note that basic properties are hereditary. A couple of important types are $\mathbf{b}$, having a single black vertex, and $\mathbf{w}$, having a single white vertex. Then $\mathcal{P}(\mathbf{b})$ is the class of complete graphs and $\mathcal{P}(\mathbf{w})$ is the class of empty graphs. It can then be seen that $\mathcal{P}_{r,s} = \mathcal{P}(s\mathbf{b} \cup (r-s)\mathbf{w}) = \mathcal{P}(\mathbf{b})^s \times \mathcal{P}(\mathbf{w})^{r-s}$, the exponents having the natural definitions in terms of products. More generally, if $\tau$ is the disjoint union of two types $\tau_1$ and $\tau_2$ then $\mathcal{P}(\tau) = \mathcal{P}(\tau_1) \times \mathcal{P}(\tau_2)$.

When considering the probability that $G_{n,p}$ is in $\mathcal{P}(\tau)$, it becomes clear that certain partitions of $[n]$ are much more likely to produce a witness to $G_{n,p} \in \mathcal{P}(\tau)$ than other partitions. The relative proportions of the parts significantly affects the probability; this effect has been observed already in the proof of Theorem 2.2. It will be enough for us to single out only those proportions that are most likely. These considerations prompt the following four definitions.

**Definition.** Let $\tau$ be a type. A *p-template* for $\tau$ is a vector $\alpha \in [0,1]^{V(\tau)}$ which minimizes the expression

$$\begin{aligned} H_p(\tau, \alpha) = &- \sum_{v \in B(\tau)} \alpha_v^2 \log_2 p - \sum_{v \in W(\tau)} \alpha_v^2 \log_2 q \\ &- 2 \sum_{uv \in EB(\tau)} \alpha_u \alpha_v \log_2 p - 2 \sum_{uv \in EW(\tau)} \alpha_u \alpha_v \log_2 q \end{aligned}$$

subject to $\sum_{v \in V(\tau)} \alpha_v = 1$. Here $0 < p < 1$ and $q = 1 - p$, so $H_p(\tau, \alpha) \geq 0$. We define $H_p(\tau)$ to be the minimum of the above expression; that is, $H_p(\tau) = H_p(\tau, \alpha)$ where $\alpha$ is a $p$-template for $\tau$.

It transpires that a graph is most likely to appear in $\mathcal{P}(\tau)$ if it has a witnessing partition compatible, in the following sense, with a $p$-template.

**Definition.** Let $\tau$ be a type and let $\mathcal{V} = \{V_v : v \in V(\tau)\}$ be a partition of $V = \bigcup_{v \in V(\tau)} V_v$. Let $\beta \in [0,1]^{V(\tau)}$. We say that $\mathcal{V}$ is *compatible* with $\beta$, written $\mathcal{V} \sim \beta$, if $\lfloor \beta_v |G| \rfloor \leq |V_v| \leq \lceil \beta_v |G| \rceil$ for all $v \in V(\tau)$.

There are also other characteristics, apart from having a witness compatible with a $p$-template, that are common to most random members of $\mathcal{P}(\tau)$. The next definition delineates those characteristics that will be of use to us later on; specifically, they will be used to establish Lemma 6.1.

**Definition.** Let $\tau$ be a type, let $\mathcal{V} = \{V_v : v \in V(\tau)\}$ be a partition of $V = \bigcup_{v \in V(\tau)} V_v$ and let $0 < \gamma < 1$. The set $\mathcal{P}(\tau, \mathcal{V}, \gamma)$ comprises all graphs

$G \in \mathcal{P}(\tau, \mathcal{V})$ satisfying the following two conditions for each pair of vertices $u, v \in \tau$ for which $uv \notin E(\tau)$ and $V_v \neq \emptyset$:

(a) each vertex $x \in V_u$ has at least $\gamma|\tau||V|$ neighbours and at least $\gamma|\tau||V|$ non-neighbours in $V_v$, and
(b) for each pair of subsets $X \subset V_u$ and $Y \subset V_v$ with $|X| > \gamma|V|$ and $|Y| > \gamma|V|$, the induced bipartite subgraph $G[X, Y]$ is neither complete nor empty.

**Definition.** Let $\tau$ be a type, let $0 < p < 1$ and let $\alpha$ be a $p$-template for $\tau$. The $\alpha$-*constrained* property $\mathcal{P}(\tau, \alpha)$ is defined by

$$\mathcal{P}(\tau, \alpha) = \bigcup_{\mathcal{V} \sim \alpha} \mathcal{P}(\tau, \mathcal{V}, \gamma) \quad \text{where}$$
$$\gamma = \gamma(\tau, p) = \frac{1}{2|\tau|} \times \min\{p, q\} \times \min\{\alpha_v : \alpha_v > 0\}.$$

If $G \in \mathcal{P}(\tau, \mathcal{V}, \gamma)$ we say that $\mathcal{V}$ *witnesses* $G \in \mathcal{P}(\tau, \alpha)$.

Note that, clearly, $\mathcal{P}(\tau, \alpha) \subset \mathcal{P}(\tau)$. Note too that when the notation $\mathcal{P}(\tau, \alpha)$ is used it is assumed that $\alpha$ is a $p$-template for $\tau$; the dependence on $p$ is suppressed. Although $\mathcal{P}(\tau, \alpha)$ is *not* a hereditary property, it is nevertheless true that $c_{n,p}(\mathcal{P}(\tau, \alpha))$ tends to some limit $c_p(\mathcal{P}(\tau, \alpha))$ as $n \to \infty$. The next lemma includes a proof of this fact, but more importantly it justifies our belief that the property $\mathcal{P}(\tau, \alpha)$ captures the most likely members of $\mathcal{P}(\tau)$.

**Lemma 5.1.** *Let $0 < p < 1$. Let $\tau$ be a type and let $\alpha$ be a $p$-template for $\tau$. Then $c_{n,p}(\mathcal{P}(\tau, \alpha))$ tends to a limit $c_p(\mathcal{P}(\tau, \alpha))$ as $n \to \infty$. Moreover*

$$c_p(\mathcal{P}(\tau, \alpha)) = c_p(\mathcal{P}(\tau)) = H_p(\tau).$$

**Proof.** The proof is similar to that of Theorem 2.2. Given $n \in \mathbb{N}$ let $\beta \in (\mathbb{N}_0/n)^{V(\tau)}$ with $\sum_{v \in V(\tau)} \beta_v = 1$. The probability of $G_{n,p} \in \mathcal{P}(\tau)$ being witnessed by some fixed partition of $[n]$ compatible with $\beta$ is

$$P_\beta = \prod_{v \in B(\tau)} p^{\binom{\beta_v n}{2}} \prod_{v \in W(\tau)} q^{\binom{\beta_v n}{2}} \prod_{uv \in EB(\tau)} p^{\beta_u \beta_v n^2} \prod_{uv \in EW(\tau)} q^{\beta_u \beta_v n^2}$$
$$= 2^{-H_p(\tau, \beta) n^2/2} 2^{-s_\beta n/2},$$

where $0 \geq s_\beta = \sum_{v \in B} \beta_v \log_2 p + \sum_{v \in W} \beta_v \log_2 q$.

Choose, for each $n \in \mathbb{N}$, a $\beta$ which maximises $P_\beta$. Since there are at most $|\tau|^n$ ways to partition $[n]$ into $|\tau|$ sets, we see that $\mathbb{P}(G_{n,p} \in \mathcal{P}(\tau)) \leq |\tau|^n P_\beta$. There is some constant $C > 0$ such that $s_\beta > -C$ for all $\beta$. So we obtain

$$c_{n,p}(\mathcal{P}(\tau)) \binom{n}{2} \geq H_p(\tau, \beta) \frac{n^2}{2} - Cn/2 - n \log_2 |\tau|.$$

Dividing by $n^2$ and letting $n \to \infty$, and recalling that $H_p(\tau, \beta) \geq H_p(\tau)$ for all $\beta$, we obtain the inequality $c_p(\mathcal{P}(\tau)) \geq H_p(\tau)$.

Now choose, for each $n \in \mathbb{N}$, a $\beta$ such that $\lfloor \alpha_v n \rfloor \leq \beta_v n \leq \lceil \alpha_v n \rceil$ for each $v$. Then $\beta \to \alpha$ as $n \to \infty$, so $\lim_{n\to\infty} H_p(\tau, \beta) = H_p(\tau, \alpha) = H_p(\tau)$. The probability that a fixed partition of $[n]$, compatible with $\beta$, witnesses $G_{n,p} \in \mathcal{P}(\tau, \alpha)$ is $P_\beta$, as stated above. The probability of conditions (a) and (b) of the definition of $G_{n,p} \in \mathcal{P}(\tau, \mathcal{V}, \gamma)$ being also satisfied is independent of $P_\beta$, and moreover it is large. Indeed, by standard estimates of the binomial distribution (see, for example, [4, Chapter I]), the probability that some given vertex $x \in V_u$ has fewer than $\gamma|\tau||V|$ neighbours or non-neighbours in $V_v$ is, by the definition of $\gamma$, exponentially small in $n$ (we omit the details, which are elementary and entirely routine); since there are only polynomially many choices for $u$, $v$ and $x$, condition (a) is almost surely satisfied. In condition (b) the number of choices for $X$ and $Y$ is exponentially large in $n$ but the probability of failure in any instance is exponentially small in $n^2$; hence condition (b) is almost surely satisfied also. So it is certainly true that, when $n$ is large, we have $\mathbb{P}(G_{n,p} \in \mathcal{P}(\tau)) \geq \mathbb{P}(G_{n,p} \in \mathcal{P}(\tau, \alpha)) \geq P_\beta/2$. Hence for large $n$,

$$c_{n,p}(\mathcal{P}(\tau))\binom{n}{2} \leq c_{n,p}(\mathcal{P}(\tau, \alpha))\binom{n}{2} \leq H_p(\tau, \beta)\frac{n^2}{2}.$$

Dividing the above inequality by $n^2$ and letting $n \to \infty$ we obtain

$$\begin{aligned} c_p(\mathcal{P}(\tau)) &\leq \liminf c_{n,p}(\mathcal{P}(\tau, \alpha)) \\ &\leq \limsup c_{n,p}(\mathcal{P}(\tau, \alpha)) \leq \lim H_p(\tau, \beta) = H_p(\tau). \end{aligned}$$

But we have already shown that $c_p(\mathcal{P}(\tau)) \geq H_p(\tau)$, so the proof is complete. ∎

## 5.2. Core properties.

When $\tau'$ is a subtype of the type $\tau$ then $\mathcal{P}(\tau') \subset \mathcal{P}(\tau)$, so $\mathbb{P}(G_{n,p} \in \mathcal{P}(\tau')) \leq \mathbb{P}(G_{n,p} \in \mathcal{P}(\tau))$ and $c_p(\mathcal{P}(\tau')) \geq c_p(\mathcal{P}(\tau))$. If $c_p(\mathcal{P}(\tau')) = c_p(\mathcal{P}(\tau))$ then $\mathbb{P}(G_{n,p} \in \mathcal{P}(\tau')) \approx \mathbb{P}(G_{n,p} \in \mathcal{P}(\tau))$, and rather than work with $\tau$ we may as well restrict our attention to $\tau'$. This prompts the following definition.

**Definition.** A type $\tau$ is a *p-core* type if, for every proper subtype $\tau'$ of $\tau$, we have $c_p(\mathcal{P}(\tau')) > c_p(\mathcal{P}(\tau))$. Thus, by Lemma 5.1, the type $\tau$ is $p$-core if and only if every $p$-template $\alpha$ has support $V(\tau)$; that is, $\alpha_v \neq 0$ for all $v \in V(\tau)$. A *p-core* property is a property $\mathcal{P}(\tau)$ where $\tau$ is a $p$-core type. We write *core* instead of $p$-core where the value of $p$ is understood.

It should be noticed that whether a type is a core type *depends on the value of* $p$. For example, let $\tau$ consist of two black vertices $u$ and $v$ plus the white edge $uv$. Then

$$H_p(\tau,\alpha) = -(\alpha_u+\alpha_v)\log_2 p - 2\alpha_u\alpha_v \log_2 q = -\log_2 p - 2\alpha_u\alpha_v(\log_2 q - \log_2 p).$$

If $p<q$ there is a unique $p$-template, namely $(1/2,1/2)$, and so $\tau$ is $p$-core. If $p>q$ there are two $p$-templates, $(1,0)$ and $(0,1)$, so $\tau$ is not $p$-core. It $p=q=1/2$ then every $\alpha$, including $(1,0)$ and $(0,1)$, is a $p$-template, so $\tau$ is not $1/2$-core.

An elementary consequence of a type being $p$-core is described by the next definition and lemma.

**Definition.** Two vertices $u$ and $v$ of a type are *similar* if they have the same colour, the edge $uv$ exists and has the same colour as $u$ and $v$ and, for every other vertex $w$, if $w$ is joined to one of $u$ and $v$ then it is joined to the other with $uw$ being the same colour as $vw$.

**Lemma 5.2.** *No two vertices of a core type are similar.*

**Proof.** Suppose that the $p$-core type $\tau$ contains two similar vertices $u$ and $v$. Let $\alpha$ be a $p$-template for $\tau$ and define $\alpha'$ by $\alpha'_u = \alpha_u+\alpha_v$, $\alpha'_v = 0$ and $\alpha'_w=\alpha_w$ for $w \neq u,v$. Then $H_p(\tau,\alpha')=H_p(\tau,\alpha)$, so $\alpha'$ is a $p$-template for $\tau$. But the support of $\alpha'$ is not $V(\tau)$, contradicting $\tau$ being $p$-core. ∎

The real significance for us of core types is as follows. If $\tau$ is a $p$-core and $\alpha$ is a $p$-template, then there is a sense in which all parts of a graph $G$ contribute uniformly to the probability $\mathbb{P}(G\in\mathcal{P}(\tau,\alpha))$. To make this vague remark precise, we require the following definition.

**Definition.** Let $\tau$ be a type and let $\beta\in[0,1]^{V(\tau)}$. For $v\in V(\tau)$, the *$\beta$-degree* $d_\beta(v)$ of $v$ is defined to be

$$d_\beta(v) = -\beta_v\mathbb{B}_v\log_2 p - \beta_v\mathbb{W}_v\log_2 q - \sum_{uv\in EB(\tau)}\beta_u\log_2 p - \sum_{uv\in EW(\tau)}\beta_u\log_2 q,$$

where $\mathbb{B}_v$ and $\mathbb{W}_v$ are the indicators of the events $(v\in B(\tau))$ and $(v\in W(\tau))$.

Observe that $\sum_{v\in V(\tau)}\beta_v d_\beta(v)=H_p(\tau,\beta)$. The next lemma shows that, if $\tau$ is a core type and $\alpha$ is a $p$-template, then the $\alpha$-degrees are independent of $v$.

**Lemma 5.3.** *Let $0<p<1$ and let $\tau$ be a $p$-core type. Let $\alpha$ be a $p$-template for $\tau$. Then $d_\alpha(v)=c_p(\mathcal{P}(\tau))$ for every $v\in V(\tau)$.*

**Proof.** As we have already remarked, Lemma 5.1 shows that the $p$-template $\alpha$ has support $V(\tau)$ because $\tau$ is $p$-core. If $\tau = \mathbf{b}$ or $\tau = \mathbf{w}$ then $\alpha = (1)$ and the lemma is trivial. Apart from these two exceptions we have $0 < \alpha_v < 1$ for each $v \in V(\tau)$. Now $\alpha$ minimises $H_p(\tau, \alpha)$ subject to the constraint $\sum_{v\in V(\tau)} \alpha_v = 1$, so it must be that $\partial H_p(\tau,\alpha)/\partial\alpha_v$ is a constant $\mu$ independent of $v$. This means that

$$\begin{aligned}\mu &= \frac{\partial H_p(\tau,\alpha)}{\partial \alpha_v}\\ &= -2\alpha_v \mathbb{B}_v \log_2 p - 2\alpha_v \mathbb{W}_v \log_2 q - 2\sum_{uv\in EB(\tau)} \alpha_u \log_2 p - 2\sum_{uv\in EW(\tau)} \alpha_u \log_2 q\\ &= 2d_\alpha(v).\end{aligned}$$

Hence $H_p(\tau,\alpha) = \sum_{v\in V(\tau)} \alpha_v d_\alpha(v) = (\mu/2)\sum_{v\in V(\tau)} \alpha_v = \mu/2$. Thus $d_\alpha(v) = H_p(\tau,\alpha)$ for every $v \in V(\tau)$. By Lemma 5.1 $H_p(\tau,\alpha) = c_p(\mathcal{P}(\tau))$, and this completes the proof. ∎

The lemma above is crucial to the argument of Lemma 6.2, which lies at the heart of our calculation of the variance of the random variable $Y$, as discussed in §4.

### 5.3. Close approximation by basic properties

We conclude this section by describing the example, mentioned earlier, of a property $\mathcal{P}$ which cannot be approximated for $p \neq 1/2$ quite as well as it can be for $p = 1/2$. We recall that, when $p = 1/2$, there is a $\tau$ such that $\mathcal{P}(\tau) \subset \mathcal{P}$ and $c_p(\mathcal{P}(\tau)) = c_p(\mathcal{P})$, namely $\tau = s\mathbf{b} \cup (r-s)\mathbf{w}$ and $\mathcal{P}(\tau) = \mathcal{P}_{r,s}$ for some appropriate $r$ and $s$. However, when $p \neq 1/2$ there may be no such $\tau$. Here is an explicit example.

Let $\mathcal{P}$ be the property of being a union of complete graphs. (This will work if $p < q$; if $p > q$ take the property of being complete multipartite, and interchange the roles of black and white in the subsequent discussion.) Let $\sigma_k$ be the type of order $k$ that is a complete graph in which the vertices are coloured black and the edges are coloured white. Then $\mathcal{P} = \bigcup_{k\in\mathbb{N}} \mathcal{P}(\sigma_k)$. Now $\sigma_k \subset \sigma_{k+1}$, $\mathcal{P}(\sigma_k) \subset \mathcal{P}(\sigma_{k+1})$ and $c_p(\mathcal{P}(\sigma_k))$ is a decreasing sequence. It is then easily seen that $c_p(\mathcal{P}) = \lim_k c_p(\mathcal{P}(\sigma_k))$. To compute $c_p(\mathcal{P}(\sigma_k))$, let $\alpha \in [0,1]^{V(\sigma_k)}$ satisfy $\sum_{v\in V(\sigma_k)} \alpha_v = 1$. Since $p < q$ the constant $c$ defined by $1 - c = (\log q)/(\log p)$ is positive. Then it is readily seen that

$$\frac{H_p(\sigma_k,\alpha)}{\log_2(1/p)} = 1 - 2c \sum_{uv\in E(\sigma_k)} \alpha_u\alpha_v \geq 1 - 2c\binom{k}{2}\frac{1}{k^2} = 1 - c + \frac{c}{k},$$

the unique $p$-template being the vector giving equal weight to every vertex. Therefore $\sigma_k$ is $p$-core, $c_p(\mathcal{P}(\sigma_k)) = (1 - c + c/k)\log_2(1/p)$ and $c_p(\mathcal{P}) = (1 - c)\log_2(1/p)$.

Suppose now that $\mathcal{P}(\tau) \subset \mathcal{P}$ for some type $\tau$ and that $c_p(\mathcal{P}(\tau)) = c_p(\mathcal{P})$. We may suppose further that $\tau$ is $p$-core, since otherwise we can replace $\tau$ by a subtype of itself. Now $\mathcal{P}$ contains only unions of complete graphs, which each have the property that any two adjacent vertices have exactly the same set of neighbours. It follows that $\tau$ has no missing edges — it is a complete graph. If $uv$ were a black edge of $\tau$ then both $u$ and $v$ would be black, so $u$ and $v$ would be similar vertices and Lemma 5.2 would be contradicted. Thus the edges of $\tau$ are white. Now it is easily seen that if $\tau$ is complete and has a white vertex then a vector $\alpha$ that gives weight one to this vertex and zero to all the other vertices is a $p$-template when $p < q$, so $\tau$ would not in fact be $p$-core. We conclude that $\tau = \sigma_k$ for some $k$, contradicting $c_p(\mathcal{P}(\tau)) = c_p(\mathcal{P})$.

## 6. Colouring by hereditary properties

The truth of Theorem 1.2 for the properties $\mathcal{P}(\mathbf{b})$ and $\mathcal{P}(\mathbf{w})$ is precisely the result of [5]. Corollary 2.1 then implies that Theorem 1.2 holds for the properties $\mathcal{P}_{r,s} = \mathcal{P}(\mathbf{b})^s \times \mathcal{P}(\mathbf{w})^{r-s}$. As remarked earlier, this implies the truth of the theorem for every $\mathcal{P}$ when $p = 1/2$. To prove the theorem in general for $p \neq 1/2$ it is enough, by Theorem 1.1, to prove it for all basic properties; in other words, we must extend the result of [5] to basic properties $\mathcal{P}(\tau)$.

To estimate the second moment of the number of $\mathcal{P}$-graphs of order $t$ induced in $G_{n,p}$ we must compute the probability that two overlapping subgraphs are both $\mathcal{P}$-graphs. Thus one of the main tools in our proof is Lemma 6.2, which gives a bound on this probability. The second moment of a reducible property behaves differently to that of an irreducible property, so we shall take $\mathcal{P}$ to be irreducible. This, it turns out, will give us a handle on pairs of $\mathcal{P}$-subgraphs that overlap by a moderate amount. In order to analyse highly overlapping $\mathcal{P}$-subgraphs we need precise information as to the edges incident with each vertex, and more generally we need constraints on the structure of these $\mathcal{P}$-subgraphs. For this reason we shall consider only constrained properties $\mathcal{P}(\tau, \alpha)$ where $\tau$ is $p$-core. The implication of $\tau$ being $p$-core is that the ways in which two $\mathcal{P}(\tau, \alpha)$-subgraphs can overlap are very limited. This statement requires a straightforward but slightly lengthy verification, so we give it in a separate lemma.

**Lemma 6.1.** *Let $0 < p < 1$, let $\tau$ be a $p$-core type and let $\alpha$ be a $p$-template for $\tau$. Let $G$ and $G'$ be $\mathcal{P}(\tau, \alpha)$-graphs with $V(G) = V(G') = T$, say. Let $\mathcal{T}$ and*

*$\mathcal{T}'$ be partitions of $T$ witnessing $G\in\mathcal{P}(\tau,\alpha)$ and $G'\in\mathcal{P}(\tau,\alpha)$ respectively and let $\gamma|T|\geq 2$ where $\gamma=\gamma(\tau,\alpha,p)$. Suppose that $G[S]=G'[S]$ where $S\subset T$ and $|S|\geq(1-\gamma)|T|$. Then $\mathcal{T}$ and $\mathcal{T}'$ induce the same partition of $S$ (to within a relabelling of the parts).*

**Proof.** Let $\mathcal{T}=\{T_v : v\in V(\tau)\}$ and $\mathcal{T}'=\{T'_v : v\in V(\tau)\}$. Given a vertex $u\in\tau$ let $S_u=T_u\cap S$ and $S'_u=T'_u\cap S$. Let $t=|T|$, and recall that $\alpha_v>0$ for all $v\in\tau$ because $\tau$ is $p$-core. Notice that, for each $u\in\tau$, we have

$$|S_u|\geq\min\{\alpha_v \;:\; v\in\tau\}\times t-1-\gamma t\geq 4|\tau|\gamma t-1-\gamma t>\gamma|\tau|t\geq 2.$$

Suppose that $a,b\in\tau$ are distinct vertices and that $A=S_u\cap S'_a$ and $B=S_u\cap S'_b$ satisfy $|A|\geq\gamma t$ and $|B|\geq\gamma t$. For the sake of definiteness let us suppose for the time being that $u$ is white. Since $|A|\geq 2$ and $G[S_u]$ is empty, $G[S'_a]$ must be empty also and so $a$ must be white; likewise $b$ must be white. Moreover, there are no edges between $A$ and $B$, so by condition (b) of the definition of $\mathcal{T}'$ witnessing $G'\in\mathcal{P}(\tau,\alpha)$, the edge $ab$ must exist and be white. Now let $c\in\tau$ be distinct from $a$ and $b$. There exists a vertex $v\in\tau$ such that $C=|S_v\cap S'_c|\geq|S'_c|/|\tau|\geq\gamma t$. If $v=u$ then there are no edges between $C$ and $A\cup B$ so, as just described, both $ac$ and $bc$ must be white edges of $\tau$. If $v\neq u$ but one of $ac$ and $bc$ exists, say $ac$, then the bipartite subgraph $G[A,C]$ is empty or complete according as $ac$ is white or black. Thus, by the definition of $\mathcal{T}$, the edge $uv$ must exist and have the same colour as $ac$. But then the bipartite subgraph $G[B,C]$ is empty or complete according to this colour, and so by the definition of $\mathcal{T}'$ the edge $bc$ must exist and have the same colour as $ac$. The upshot of this argument is that $a$ and $b$ are similar vertices of $\tau$, which is impossible by Lemma 5.2. Moreover, the same contradiction follows under the assumption that $u$ is black, as can easily be seen.

We conclude, therefore, that for each vertex $u\in\tau$ there is only one vertex $u^*\in\tau$ such that $|S_u\cap S'_{u^*}|\geq\gamma t$, and since $|S_u|\geq\gamma|\tau|t$ there must be exactly one. This naturally defines a map $u\mapsto u^*$ from $V(\tau)$ to itself. If $u^*=v^*$ then $|S_u\cap S'_{u^*}|\geq\gamma t$ and $|S_v\cap S'_{u^*}|\geq\gamma t$, so by the previous argument for $S_u$ applied instead to $S'_{u^*}$ we conclude that $u=v$. So the map $u\mapsto u^*$ is a bijection. Now the colours of $u$ and $u^*$ must be the same because $|S_u\cap S'_{u^*}|\geq 2$ and, what is more, if $uv$ is an edge then, by condition (b), the edge $u^*v^*$ must exist and be the same colour as $uv$. Therefore the map $u\mapsto u^*$ is an automorphism of $\tau$. However, it may not give an automorphism of $\alpha$ so we cannot just assume that the map is the identity.

Suppose now that $S'_{u^*}\not\subset S_u$. Then there is some vertex $x\in S'_{u^*}\cap S_v$ where $v\neq u$. Once again, we shall suppose further that $u$ is white, in which case so is $u^*$ because $u\mapsto u^*$ is an automorphism. Now there are no edges from $x\in S'_{u^*}$ to $S'_{u^*}\cap S_u$, but $x\in S_v$ and $|S_u\setminus S'_{u^*}|=|S_u\cap\bigcup_{v\neq u^*}S'_v|<\gamma|\tau|t$. So condition (a) of the definition of $\mathcal{T}$ witnessing $G\in\mathcal{P}(\tau,\alpha)$ shows that $uv$ must be in $E(\tau)$ and must be white. Then $u^*v^*$ exists and is white also, so there are no edges from $x\in S'_{u^*}$ to $S'_{v^*}\cap S_v\neq\emptyset$. As $x\in S_v$ this means that $v$ is white. Now let $w$ be a vertex of $\tau$ distinct from $u$ and $v$. If $uw$ exists, so does $u^*w^*$ with the same colour, so all or none of the edges from $x\in S'_{u^*}$ to $S'_{w^*}$ exist, according as $uw$ is black or white. Since $x\in S_v$ and $|S_w\setminus S'_{w^*}|<\gamma|\tau|t$, condition (a) of the definition of $\mathcal{T}$ shows that the edge $vw$ must exist and be the same colour as $uw$. If, on the other hand, we begin from the assumption that $vw$ exists, then all or none of the edges from $x\in S_v$ to $S_w$ exist, according as $vw$ is black or white. Since $x\in S'_{u^*}$ and $|S'_{w^*}\setminus S_w|<\gamma|\tau|t$, condition (a) of the definition of $\mathcal{T}'$ lets us deduce that $u^*w^*$ exists and is the same colour as $vw$, in which case $uw$ also exists and has this colour. So we conclude that $u$ and $v$ are similar vertices of $\tau$, an impossibilty by Lemma 5.2. Once again, the same contradiction follows from the assumption that $u$ is black.

Finally, we know that $S'_{u^*}\subset S_u$ for every $u\in\tau$. But, likewise, $S_u\subset S'_{u^*}$; in other words, $S_u=S'_{u^*}$ for all $u\in\tau$. ∎

We are now in a position to prove the main lemma of this section.

**Lemma 6.2.** *Let $G=G_{n,p}$ be a random graph with vertex set $[n]=\{1,\dots,n\}$. Let $\tau$ be an irreducible $p$-core type and let $\alpha$ be a $p$-template for $\tau$. Then there exists a positive constant $c=c(\tau,\alpha,p)$ such that for all $0<\delta<1$ the following holds, provided $t$ is sufficiently large. Let $S\subset T\subset[n]$ with $|S|=s$ and $|T|=t$, where $\delta t\le s\le t$. Then*

$$\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\mid G[S]\right\}\le 2^{-(c_p/2+c\delta)t(t-s)},$$

*where $c_p=c_p(\mathcal{P}(\tau))$.*

**Proof.** The event $G[T]\in\mathcal{P}(\tau,\alpha)$ occurs only if $\tau$ is witnessed by some partition $\mathcal{T}$ of $T$, say $\mathcal{T}=\bigcup_{v\in V(\tau)}T_v$ with $|T_v|\ge\lfloor\alpha_v t\rfloor\ge\alpha_v t-1$ for each $v\in\tau$. Given some fixed partition $\mathcal{T}$, we can regard the random subgraph $G[T]$ as being generated in the usual way, one experiment being performed for each edge to determine its presence. In this case, the probability (denoted $P_x$) of the edges/non-edges arising at a given vertex $x\in T_v$ that are necessary for $\mathcal{T}$ to witness $\tau$ is

$$P_x=p^{(|T_v|-1)\mathbb{B}_v}q^{(|T_v|-1)\mathbb{W}_v}\prod_{uv\in EB(\tau)}p^{|T_u|}\prod_{uv\in EW(\tau)}q^{|T_u|}\le R^{|\tau|+1}2^{-d_\alpha(v)t},$$

where $R=\max\{1/p,1/q\}$ and $d_\alpha(v)$ is the $\alpha$-degree of $v$ as defined in §5.2. By Lemma 5.3, $d_\alpha(v)=c_p$ for every $v\in\tau$. So, writing $C=R^{|\tau|+1}$, we see that the probability that the edges at $x$ are consistent with $\mathcal{T}$ witnessing $\tau$ is at most $C2^{-c_pt}$.

On the other hand, we can regard the random subgraph $G[T]$ as being generated not by $\binom{t}{2}$ experiments but by $t(t-1)$ experiments, as follows. If $x,y\in T$ and the edge $xy$ is required to be present in order for $\mathcal{T}$ to witness $\tau$, we perform two independent experiments each with probability $\sqrt{p}$ of success, one *associated* with $x$ and the other with $y$; the edge is inserted only if both experiments succeed. Likewise if an edge is required to be absent we perform two experiments each with probability $\sqrt{q}$. Thus the partition $\mathcal{T}$ will witness $\tau$ only if every experiment at every vertex succeeds. Let $x\in T_v$. The probability that the experiments associated with $x$ succeed is

$$\sqrt{p}^{(|T_v|-1)\mathbb{B}_v}\sqrt{q}^{(|T_v|-1)\mathbb{W}_v}\prod_{uv\in EB(\tau)}\sqrt{p}^{|T_u|}\prod_{uv\in EW(\tau)}\sqrt{q}^{|T_u|}=\sqrt{P_x}\le\sqrt{C}2^{-c_pt/2},$$

where $C$ is as defined above.

Now suppose that the edges of $G[S]$ have already been determined. We say that the partition $\mathcal{T}=\bigcup_{v\in V(\tau)}T_v$ of $T$ is *consistent* with $G[S]$ if $\mathcal{T}$ witnesses some $\mathcal{P}(\tau,\alpha)$-graph on the vertex set $T$ that coincides on the subset $S$ with $G[S]$. If there is no partition $\mathcal{T}$ consistent with $G[S]$ then the conditional probability that we are seeking to estimate is zero, and the lemma holds trivially. So let us assume that $\mathcal{T}$ is a partition consistent with $G[S]$.

There are two ways that we can estimate the probability that $\mathcal{T}$ witnesses $\tau$ conditional on it being consistent with a given $G[S]$. Denote by $P(T-S)$ the probability that the edges of $G[T-S]$ are consistent with this witness. Using the single experiment model of the first paragraph,

$$\mathbb{P}\{\mathcal{T}\text{ witnesses }\tau\mid G[S]\}=P(T-S)^{-1}\prod_{x\in T-S}P_x\le R^{\binom{t-s}{2}}\,C^{t-s}\,2^{-c_pt(t-s)}.$$

On the other hand, using the double experiment model, we see that this probability is the probability that all experiments at all vertices $x\in T-S$ succeed and that an extra experiment succeeds for each required edge/non-edge between $S$ and $T-S$. Let $a$ be the number of such required edges, $b$ the number of required non-edges and let $e_{\mathcal{T}}=a+b$. Writing $r=\max\{\sqrt{p},\sqrt{q}\}$, we have

$$\mathbb{P}\{\mathcal{T}\text{ witnesses }\tau\mid G[S]\}=\sqrt{p}^a\sqrt{q}^b\prod_{x\in T-S}\sqrt{P_x}\le r^{e_{\mathcal{T}}}\,C^{(t-s)/2}\,2^{-c_pt(t-s)/2}.$$

Suppose first that $s$ is large: to be precise, $t-s\le\gamma t$ where $\gamma=\gamma(\tau,\alpha,p)$. Then if $t$ is large enough we have $\gamma t\ge 2$ and the conditions of Lemma 6.1 are satisfied. So in this case, if $G[S]$ is given, all partitions $\mathcal{T}$ of $T$ consistent with $G[S]$ induce the same partition of $S$ (to within a relabelling of the parts). Consequently, there are at most $|\tau|! \times |\tau|^{t-s}$ partitions $\mathcal{T}$ consistent with the given $G[S]$. Therefore

$$\begin{aligned}&\mathbb{P}\{G[T] \in \mathcal{P}(\tau,\alpha) \mid G[S]\}\\&\qquad \leq |\tau|!\,|\tau|^{t-s}\, R^{\binom{t-s}{2}}\, C^{t-s}\, 2^{-c_p t(t-s)} \leq 2^{(-c_p t + A(t-s))(t-s)}\end{aligned}$$

for some constant $A = A(\tau, p)$. Let $\nu = \min\{\gamma, c_p/4A\}$. It follows that if $0 \leq t - s \leq \nu t$ and $t$ is sufficiently large then $\mathbb{P}\{G[T] \in \mathcal{P}(\tau,\alpha) \mid G[S]\} \leq 2^{-3c_p t(t-s)/4}$, in which case the lemma certainly holds.

All that remains is to verify the lemma in the case where $\delta t \leq s < (1-\nu)t$, where $\nu$ is as in the previous paragraph and $0 < \delta < 1$ is given. We may, and we shall, assume that $\delta < \alpha_0/4$, where $\alpha_0 = \min\{\alpha_v : v \in \tau\}$. We shall show that $e_{\mathcal{T}}$, the number of required edges/non-edges between $S$ and $T - S$, must be of order $\delta t^2$. Let $\mathcal{T}$ be a partition witnessing $\tau$, and let $U = \{u \in \tau : |T_u \cap S| < \delta t/|\tau|\}$. Then $U \neq V(\tau)$. If $U \neq \emptyset$ we may, by the irreducibility of $\tau$, select $u \in U$ and $v \notin U$ such that $uv$ is an edge of $\tau$. Then, since $|T_u| > \alpha_0 t/2$, we have

$$e_{\mathcal{T}} \geq |T_v \cap S| \times |T_u \cap (T - S)| \geq \frac{\delta t}{|\tau|} \cdot \left(\frac{\alpha_0}{2} - \frac{\delta}{|\tau|}\right) t > \frac{\alpha_0}{4|\tau|} \delta t^2.$$

On the other hand, if $U = \emptyset$, select $u$ with $|T_u \cap (T - S)| \geq |T - S|/|\tau|$; of course, such a $u$ must exist. Then

$$e_{\mathcal{T}} \geq |T_u \cap S| \times |T_u \cap (T - S)| \geq \frac{\delta t}{|\tau|} \cdot \frac{t-s}{|\tau|} > \frac{\nu}{|\tau|^2} \delta t^2.$$

So we find that there is a constant $\lambda = \lambda(\tau, \alpha, p) > 0$ such that, in each case, $e_{\mathcal{T}} \geq \lambda \delta t^2$.

Let $c = (\lambda/2)\log_2(1/r)$; then $c > 0$ because $r = \max\{\sqrt{p}, \sqrt{q}\} < 1$. Now there are at most $|\tau|^t$ partitions $\mathcal{T}$ of $T$. So, for $\delta t \leq s < (1-\nu)t$, we have

$$\begin{aligned}&\mathbb{P}\{G[T] \in \mathcal{P}(\tau,\alpha) \mid G[S]\}\\&\qquad \leq |\tau|^t\, r^{e_{\mathcal{T}}}\, C^{(t-s)/2}\, 2^{-c_p t(t-s)/2} \leq 2^{-(c_p(t-s)/2 - B + 2c\delta t)t},\end{aligned}$$

where $B = B(\tau, p)$ is some constant. Consequently, provided $t$ is large enough, we have that $\mathbb{P}\{G[T] \in \mathcal{P}(\tau,\alpha) \mid G[S]\} \leq 2^{-(c_p/2 + c\delta)t(t-s)}$, as desired. ∎

The main result of this section establishes Theorem 1.2 for basic properties.

**Theorem 6.1.** *Theorem 1.2 holds for basic properties $\mathcal{P} = \mathcal{P}(\tau)$.*

**Proof.** By the definition of a $p$-core type and by Lemma 1.1, we may assume that $\tau$ is $p$-core (for otherwise, replace $\tau$ by a $p$-core type $\tau' \subset \tau$). By Corollary 2.1 we need to prove the lemma only for the irreducible factors of $\tau$; since the factors of a $p$-core property must also be $p$-core, we may assume that $\tau$ is both $p$-core and irreducible, and so Lemma 6.2 applies to $\tau$.

First we define some constants. We write $c_p$ for $c_p(\mathcal{P})$ and $R$ for $\max\{1/p, 1/q\}$. Let $\delta = \min\{(c_p/3)\log_2 R, 1/2\}$. Let $\alpha$ be a $p$-template for $\tau$ and let $c = c(\tau, \alpha, p)$ be the constant described by Lemma 6.2. Let $\epsilon_0 = \min\{c\delta/2, 1/5c_p\}$.

As remarked in §1, we shall imitate the proof in [5]. The proof is based on the simple trick of considering the random variable $X = X(G_{n,p})$ counting the maximal number of vertex subsets of order $t$ (to be defined), the subsets each inducing a $\mathcal{P}$-graph and pairwise having no more than one vertex in common. The value of $X$ changes by at most one if we change an edge of $G_{n,p}$ to a non-edge, or vice-versa, and we may apply the Azuma-Hoeffding inequality (see [12]) to show that $X$ is sharply concentrated near its mean. It is the estimation of the mean of $X$ that requires work; we achieve it by relating $X$ to a random variable $Y$ which is more easily analysed.

Let $0 < \epsilon < \epsilon_0$ and $t = \lfloor (2/c_p) - \epsilon) \log_2 n \rfloor$. Given this value of $t$, the random variable $X$ is defined as above. We claim that $\mathbb{E}(X)$, the expected value of $X$, is at least $n^{5/3}$ if $n$ is large. The proof of the lemma will follow from this claim for, by the martingale inequalities of Hoeffding or of Azuma, for every $u \geq 0$ the inequality $\mathbb{P}\{|X - \mathbb{E}(X)| \geq u\} \leq 2\exp(-2u^2/\binom{n}{2})$ holds. Taking $u = \mathbb{E}(X)$ we see that $\mathbb{P}\{X = 0\} \leq 2^{-n^{4/3}}$. Hence the conditions of Lemma 4.1 hold for property $\mathcal{P}$ and so Theorem 1.2 does also.

To prove our claim, denote by $G_{m,p}$ the subgraph of $G_{n,p}$ induced on the vertex set $[m] = \{1, \ldots, m\} \subset V(G_{n,p})$. Let $Y_m = Y_m(G_{n,p})$ be the number of induced subgraphs of $G_{m,p}$ of order $t$ having the constrained property $\mathcal{P}(\tau, \alpha)$. We shall show that, for some $m$, $\mathbb{E}(Y_m)$ is close to $2n^{5/3}$ and that the variance of $Y_m$ is small; this will imply that $\mathbb{E}(X)$ is also at least $n^{5/3}$ and will establish our claim. Precisely, $\mathbb{E}(Y_m) = \binom{m}{t} 2^{-c_{t,p}(\mathcal{P}(\tau,\alpha))\binom{t}{2}}$. Since $\epsilon < 1/5c_p$, Lemma 5.1 tells us that if $n$ is large enough we have $7/4c_{t,p}(\mathcal{P}(\tau, \alpha)) + \eta < t/\log_2 n < 2/c_{t,p}(\mathcal{P}(\tau, \alpha)) - \eta$ for some $\eta > 0$. So, writing $m_0 = \lceil n^{7/8} \rceil$, we obtain

$$\begin{aligned}
\mathbb{E}(Y_{m_0}) &< \left( m_0 2^{-c_{t,p}(\mathcal{P}(\tau,\alpha))t/2} \right)^t \\
&< 1 < n^2 \\
&< \left( (n/t) 2^{-c_{t,p}(\mathcal{P}(\tau,\alpha))t/2} \right)^t < \mathbb{E}(Y_n)
\end{aligned}$$

if $n$ is large. Now $\mathbb{E}(Y_m)$ increases monotonically with $m$, but slowly; certainly $\mathbb{E}(Y_{m+1})<2\mathbb{E}(Y_m)$. It follows that there exists some $m$, $m_0<m<n$, for which $2n^{5/3}<\mathbb{E}(Y_m)<4n^{5/3}$. Let us fix such an $m$.

For each $T\subset V(G_{m,p})$ with $|T|=t$ let $I_T$ be the indicator variable for the event $\{G_{m,p}[T]\in\mathcal{P}(\tau,\alpha)\}$; thus $Y_m=\sum_T I_T$. Let $\Delta=\sum_{T,T'} I_T I_{T'}$, the sum being over all unordered pairs $T\neq T'$ such that $|T\cap T'|\geq 2$, that is to say, over pairs such that the induced subgraphs are distinct but share at least one common edge. Given a graph $G_{m,p}$, by deleting from the list of its $\mathcal{P}$-subgraphs any sharing an edge with another $\mathcal{P}$-subgraph, we see that $G_{m,p}$ contains a set of at least $Y_m-\Delta$ edge-disjoint $\mathcal{P}(\tau,\alpha)$-subgraphs of order $t$. Since $\mathcal{P}(\tau,\alpha)\subset\mathcal{P}$ we have that $X\geq Y_m-\Delta$. We shall show, provided $n$ is large enough, that $\mathbb{E}(\Delta)<\mu/2$ where $\mu=\mathbb{E}(Y_m)$; then $\mathbb{E}(X)\geq \mathbb{E}(Y_m)-\mathbb{E}(\Delta)\geq\mu/2\geq n^{5/3}$ as claimed, so completing the proof.

Given $0\leq s\leq t$, let $p(t,s)$ be the least upper bound for the conditional probability $\mathbb{P}\{G[T]\in\mathcal{P}(\tau,\alpha)\,|\,G[S]\}$ valid for all possible graphs $G[S]$ induced by some set $S\subset T$ with $|S|=s$. Then, writing $G$ for $G_{m,p}$, we have

$$\begin{aligned}
\mathbb{E}(\Delta) &= \sum_{T,T'}\mathbb{P}(I_T I_{T'}=1)\\
&= \sum_{T,T'}\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\right\}\cap\left\{G[T']\in\mathcal{P}(\tau,\alpha)\right\}\\
&= \sum_{T}\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\right\}\times\sum_{T'}\mathbb{P}\left\{G[T']\in\mathcal{P}(\tau,\alpha)\mid G[T]\in\mathcal{P}(\tau,\alpha)\right\}\\
&\leq \sum_{T}\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\right\}\times\sum_{T'}p(t,|T'\cap T|)\\
&= \mathbb{E}(Y)\sum_{s=2}^{t}\binom{m}{t-s}\binom{t}{s}p(t,s).
\end{aligned}$$

All that remains, then, is to show that $\sum_{s=2}^t a_s\leq 1/2$ where $a_s=\binom{m}{t-s}\binom{t}{s}p(t,s)$. We shall use two different upper bounds for $a_s$, depending on whether $s$ is small or large. First of all, observe that

$$\begin{aligned}
&\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\mid G[S]\right\}\\
&\qquad=\frac{\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\text{ and }G[S]\right\}}{\mathbb{P}\left\{G[S]\right\}}\leq\frac{\mathbb{P}\left\{G[T]\in\mathcal{P}(\tau,\alpha)\right\}}{\mathbb{P}\left\{G[S]\right\}}.
\end{aligned}$$

Now $\mathbb{P}\{G[S]\}\geq R^{-\binom{s}{2}}$, where $R=\max\{1/p,1/q\}$. Also, $\mathbb{P}\{G[T]\in\mathcal{P}(\tau,\alpha)\}=\mu\binom{m}{t}^{-1}$. So we have $a_s\leq b_s=\mu\binom{m}{t}^{-1}\binom{m}{t-s}\binom{t}{s}R^{\binom{s}{2}}$. Recalling the definition of $\delta$, note that if $2\leq s\leq\delta t$ then $s\leq(2/3)\log_R n$, in which case $b_{s+1}\leq b_s/2$

because $m \geq m_0 \geq n^{7/8}$. Consequently

$$\sum_{s=2}^{\delta t} a_s \leq 2b_2 \leq \frac{Rt^2}{(m-t)^2}\mu = o(1)$$

because $\mu < 4n^{5/3} \leq 4m^{40/21}$. As for the remaining summands, we may assume that $n$ is sufficiently large for Lemma 6.2 to hold. Recalling that $\epsilon < c\delta/2$, we have

$$\begin{aligned}\sum_{s=\delta t}^{t} a_s &= \sum_{s=\delta t}^{t} \binom{m}{t-s}\binom{t}{t-s} p(t,s) \\ &\leq \sum_{s=\delta t}^{t} \left(mt2^{-(c_p/2+c\delta)t}\right)^{t-s} < \sum_{s=\delta t}^{t} \left(\frac{1}{2}\right)^{t-s}.\end{aligned}$$

It follows that $\sum_{s=\delta t}^{t} a_s = o(1)$, and the proof of the theorem is complete. ∎

As remarked in the introduction, given Theorem 1.1 it is easy to extend Theorem 6.1 to all hereditary properties and so to complete the full proof of Theorem 1.2.

**Proof of Theorem 1.2.** Let $\epsilon > 0$ and $c_p = c_p(\mathcal{P})$. By Theorem 1.1, there exists a type $\tau$ such that $c_p(\mathcal{P}(\tau)) \leq c_p + \epsilon$ and $\chi_{\mathcal{P}}(G_{n,p}) \leq \chi_{\mathcal{P}(\tau)}(G_{n,p})$. Then, by Lemma 1.1 and Theorem 6.1, we have, almost surely,

$$c_p \frac{n}{2\log_2 n}(1+o(1)) \leq \chi_{\mathcal{P}}(G_{n,p}) \leq \chi_{\mathcal{P}(\tau)}(G_{n,p}) \leq (1+\epsilon)c_p \frac{n}{2\log_2 n}(1+o(1)).$$

This holds true for all $\epsilon > 0$, implying the truth of the theorem. ∎

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